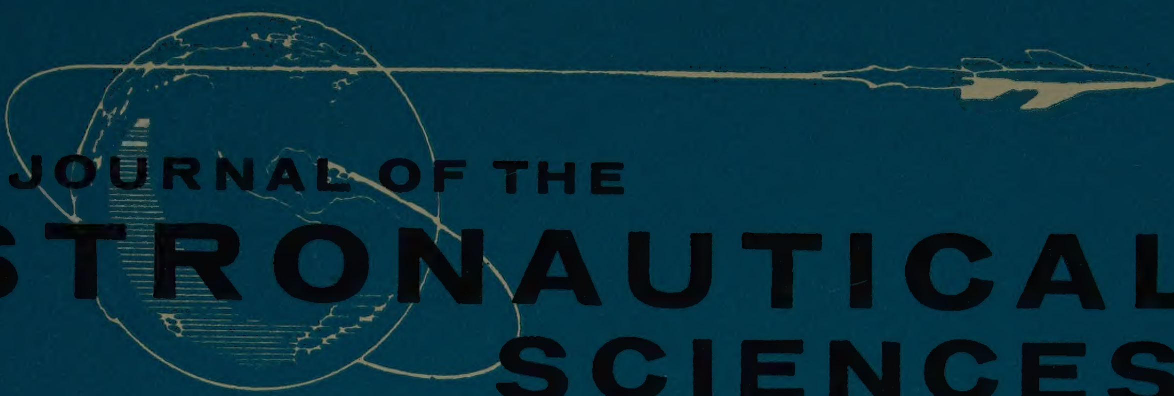

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CONTENTS

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Analysis and Standardization of Astrodynamic Constants[1]

M. W. Makemson;[2] R. M. L. Baker, Jr.;[2,3] and G. B. Westrom[4]

stract

n analysis is presented of heliocentric, geocentric, seleno-tric, planetocentric, and atmospheric astrodynamic stants. A thorough survey of past data is presented and et of adopted values for all constants and their errors is posed in order to establish a standard.

troduction

Classical astronomical constants, however precisely ermined, do not suffice for predicting trajectories space vehicles. Motions of close satellites, for ex-ple, have brought to light higher order terms in Earth's gravitational potential which have no ect on the distant Moon. For successful lunar and erplanetary missions more exact values of the solar rallax, the astronomical unit in laboratory units, d the distances, diameters, figures, masses, tem-ratures and atmospheres of the Moon and target nets are clearly essential.

The new astrodynamic constants, associated with ce vehicle trajectories, may be viewed in three ects: (a) the advance determination of the most bable values; (b) pre-knowledge of the error to be ected in the final outcome and how to correct it in ht; and (c) improvement of the constants on the sis of further observations, usually by a least squares ferential correction of the orbit. For example, the sent uncertainty in the Earth's radius may lead a 78-mile error on lunar impact as shown by Tross 0)[5] and an inaccuracy of 0″.01 in the solar parallax y produce a divergence of 50,000 miles from Mars.

An occasional review and comparison of the various erminations of astrodynamic constants is clearly a essity, and it is hoped that the adoption of con-tent values, as suggested at the end of each table the present paper, may serve to standardize compu-ions.

[1] This work was supported by the United States Air Force, part, under Contract AF 49(638)-498 monitored at UCLA the AFOSR. Presented at the Seventh Annual Meeting of American Astronautical Society January 17, 1961. Manu-ipt received September 1960, revised December 1960.

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[4] Aeronutronic Division of Ford Motor Company, Boston.

[5] Numbers in parentheses refer to references found at the clusion of the paper.

1. Heliocentric Constants

In a heliocentric orbit the solar parallax, $\Pi_\odot$, and the gravitational constant, k_s, are of first importance,

$$\Pi_\odot \text{ (radians)} = \frac{a_e}{a_\oplus} \tag{1}$$

where a_e is the Earth's equatorial radius and $a_\oplus$ its mean distance from the Sun. The astronomical unit was originally defined as the semi major axis of the Earth's orbit and was so used by Gauss, neglecting perturbations, in his classic determination of the numerical value of k_s, the "Gaussian constant." In order to preserve Gauss' value and eliminate necessary revisions of planetary tables as more accurate funda-mental constants became available, the definition of the astronomical unit was changed to "the mean distance of a fictitious unperturbed planet having the mass and sidereal period that Gauss adopted for the Earth" when he calculated k_s from Kepler's third law, i.e.,

$$k_s = \frac{2\pi a^{3/2}}{P\sqrt{1+m}} \tag{2}$$

where m is the mass of a planet in units of the Sun's mass; a its mean distance from the Sun in astronomical units, and P its sidereal period. As Gauss' value of k_s is retained for convenience by international agree-ment, it is clear that the Earth's mean distance in astronomical units (a.u.) is subject to improvement as more exact determinations of the solar parallax and the mass and sidereal period of the Earth become avail-able. Predicted planet positions, also dependent on k_s, are thus improvable up to the limits imposed by knowledge of the Earth's mass and period of revolution, subject to the errors of observation in an individual case. The general limitation is of the order of 8 or 9 significant figures. The adopted values for these funda-mental heliocentric constants are

k_s = 0.017,202,098,95 radians/mean solar day

= 3,548″.187,606,965,1 of arc/mean solar day

$a_\oplus$ = 1.000,000,03 a.u., Newcomb (76) 1895; American Ephemeris (5).

Present inaccurate values of the solar parallax greatly restrict the accuracy of the ratio of the as-

tronomical unit to laboratory units, thus downgrading the 8 or 9 figure accuracy of the astronomical system to the 4 or 5 figure significance of the ratio. Table 1 illustrates the situation. In Table 1 Rabe's value is adopted with a larger uncertainty in order to partially encompass the values recently determined by Price *et al.* (as recomputed by Westrom and Arsenault (40)) and McGuire *et al.* (will not encompass Spencer Jones' value, however). The error is chosen not only on the basis of the probable experimental errors quoted by the investigators; but also upon an estimate of systematic errors that must underlie the diversity in the results. The same philosophy will govern errors adopted in subsequent tables.

Neglecting a small uncertainty in a_e, the Earth's equatorial radius, the divergences of interplanetary trajectories corresponding to an inaccuracy in the solar parallax, expressed as the ratio

$$\Pi' = \frac{\Delta\Pi_\odot}{\Pi_\odot}\left(= \frac{0.002}{8.798} \cong 0.000{,}23 \text{ adopted 1961}\right) \quad (3)$$

are found in Table 2.

2. Geocentric Constants

Mathematical terms in the gravitational potential representing departures of the reference spheroid (approximating the Earth's figure) from a concentrically homogeneous sphere are of two types: zonal harmonics, expressing the meridional ellipticity and as[ym]metry, and tesseral or sectorial harmonics in longit[ude] which due to the Earth's rotation must be expre[ssed] as functions of the time.

The form of the Earth's gravitational poten[tial] given by Baker and Makemson (7) (Equation [...] page 90) as:

$$\Phi = \frac{k_e^2 m_1}{r}\left[1 + \frac{J}{3}\frac{1}{r^2}(1 - 3\sin^2\delta) + \frac{H}{5}\frac{1}{r^3}(3 - 5\sin^2\delta)\sin\delta + \frac{K}{30}\frac{1}{r^4}(3 - 30\sin^2\delta + 35\sin^4\delta) + \cdots\right],$$

should, probably, be modified in the light of Brouw[er's] suggestion (10) to read

$$\Phi = \frac{k_e^2 m_1}{r}\left[1 + \frac{J_2}{2r^2}(1 - 3\sin^2\delta) + \frac{J_3}{2r^3}(3 - 5\sin^2\delta)\sin\delta - \frac{J_4}{8r^4}(3 - 30\sin^2\delta + 35\sin^4\delta) - \frac{J_5}{8r^5}(15 - 70\sin^2\delta + 63\sin^4\delta)\sin\delta + \frac{J_6}{16r^6}(5 - 105\sin^2\delta + 315\sin^4\delta - 231\sin^6\delta) + \cdots\right.$$

TABLE 1

The Solar Parallax and Earth's Mean Distance from Sun

Solar Parallax[a]	Author	Method	Correspondi[ng] Distance ($\times 10^6$ km)
8″803,6 ± 0″004,6	Gill (35) 1897	triangulation on Victoria, Iris, Sappho	149.437
8.806 ± 0.004	Hinks (44) 1904	triangulation on Eros	149.40
8.803 ± 0.004	Spencer Jones (97) 1928	radial velocities	149.45
8.799 ± 0.001	Witt (107) 1935	dynamical—Eros (cf. section 3.1)	149.52
8.790 ± 0.001	Spencer Jones (98) 1941	triangulation on Eros	149.67
8.805 ± 0.007	Adams (1) 1941	radial velocities of stars	149.41
8.798,4 ± 0.000,4	E. Rabe (85) 1949	dynamical—Eros	149.526
8.792 ± 0.003	Brouwer (9) 1950	occultations[b]	149.63
8.801,84 ± 0.000,05	Price (84) 1959	radar echoes from Venus (recomputed by Hilton and Arsenault)	149.467
8.797,38 ± 0.000,1	McGuire, Morrison, Wong (69) 1960	Doppler data from Pioneer V[c]	149.54
8.797,9 ± 0.000,2	de Vaucouleurs (30) 1961	Adopted from mean of several determinations[d]	149.53
8.798 ± 0.002	Adopted (1961)		

(Newcomb's value determined in 1895 of 8″80 is adopted by the American *Ephemeris* (5), and appears in all editions.)

[a] The astronomer is concerned with practical situations and must continually endeavor to refine the fundamental constant[s on] which his theories rest. He therefore carries doubtful digits for the sake of consistency with other more precisely known const[ants] and in order to guard his numbers against end-figure error in numerical work.

[b] When P, the constant of parallactic inequality in the Moon's longitude, has been found by observation, in this case star occu[lta]tions, the solar parallax is determined by the relation

$$P = 14.566{,}2\,\frac{1 - m_{☾}}{1 + m_{☾}}.$$

In Brouwer's derivation, Hinks' value for $m_{☾}^{-1}$ was adopted.

[c] A differential correction process based on a least squares inversion has been utilized to improve the solar parallax and o[ther] astrodynamic constants by the referenced investigators.

[d] de Vaucouleurs (30) considers mean of trigonometric methods since 1895 (8″798,6 ± 0.001,5); mean of dynamical meth[ods] (8″798,3 ± 0.000,3); parallactic inequality of Moon (8″796,7 ± 0.001,8); Lunar inequality of Earth (8″798,0 ± 0.009,0); Light e[qua]tion (8″797,7 ± 0.013,2); constant of aberration, geometric (8″797,9 ± 0.001,8); constant of aberration, spectroscopic (8″797, [±] 0.002,2); and mass of Earth and Moon, Pioneer V (8″797,1 ± 0.000,8).

TABLE 2

Inaccuracies Due to Solar Parallax Uncertainty

Π′	0.0001	0.001
Mars	8,400 km	84,000 km
	5,200 mi	52,000 mi
Venus	3,500 km	35,000 km
	2,200 mi	22,000 mi.

These values when translated into correction fuel require-nts take on a considerable significance.

ere m_1 is the Earth's mass, the mass of the satellite ing negligible; r is the radius vector of the satellite's oit in units of Earth's equatorial radii, a_e, and δ declination; and the $J_2 \cdots J_6$ are the coefficients the higher zonal harmonics. Relations between J_2, and J_4, and J, H, and K are as follows:

$$J_2 = \tfrac{2}{3}J, \qquad J_3 = \tfrac{2}{5}H, \qquad J_4 = -\tfrac{4}{15}K, \qquad (7)$$

spectively (Jeffreys' $D = -\tfrac{35}{8}J_4$).

Constants involved in the Earth's gravitational ld also include

a_e the equatorial radius,

k_e the geocentric gravitational constant,

$f \triangleq \dfrac{a_e - b_e}{a_e}$, the flattening or ellipticity of the meridian,

g_e the acceleration due to gravity at the equator.

κ the coefficient of the term, $-a_e \sin^2 2\phi'$ (where ϕ' is the geocentric latitude) expressing the depression of the spheroid or equipotential surface below the reference ellipsoid, reaching a maximum at 45°. Cf. de Sitter 1924. Its numerical value is variously given between 0 and 6.8×10^{-7}. Cf. Herrick, Chapter 3 of *Astrodynamics.*

Jacchia (47), 1958, obtained the following value m a study of the motions of 1957 β_1 and 1958 β_2

$$\kappa = 3(\pm 3) \times 10^{-7}$$

nsistent with

$$1/f = 298.28 \pm 0.11$$

$$J = 0.001624 \pm 0.000001$$

$$K = 9(\pm 2) \times 10^{-7}.$$

Orbits of close satellites are subject to large secular rturbations in the node and perigee as well as to nspicuous long periodic terms in other elements otably the eccentricity) that are proving to be of eat value in more exact determinations of the J_i Equation 6. These newly determined J_i are related f, κ and $\tilde{\omega}$ $(= a_e^3\omega^2/k_e^2)$ as follows:

$$J_2 = \tfrac{2}{3}f - \tfrac{1}{3}f^2 - \tfrac{1}{3}\tilde{\omega} + \tfrac{3}{7}f\tilde{\omega} + \tfrac{8}{21}\kappa$$

$$J_4 = -\tfrac{4}{5}f^2 + \tfrac{4}{7}f\tilde{\omega} - \tfrac{32}{35}\kappa$$

eliminating κ,

$$f^2 - (1 + \tilde{\omega})f + \tfrac{1}{2}\tilde{\omega} + \tfrac{3}{2}J_2 + \tfrac{5}{8}J_4 = 0 \qquad (8)$$

where the values for ω, the Earth's angular velocity, and $\tilde{\omega}$ are

$$\omega^2 = 5{,}317.49 \times 10^{-12} \text{ radians}^2/\text{sec}^2$$

$$\tilde{\omega} = 0.003{,}461{,}39$$

corresponding to the value $a_e = 6{,}378{,}150$ meters (cf. table 5).

In Tables 3 and 4 are listed determinations of $1/f$ and $J_2 \cdots J_6$ coefficients of the higher harmonics; Table 5 shows values of a_e; while Table 6 presents a summary of the adopted geocentric constants. Note that seventh and higher order harmonics as well as longitude dependent terms have been omitted. As more data become available values of these astrodynamic constants should be adopted as well. For a discussion of these constants see Michielsen (72) and Kaula (52). In table 6 the values of k_e and g_e are chosen to be consistent with the adopted values of a_e and $1/f$; and $J_2 \cdots J_5$ are consistent among themselves and with $\dfrac{1}{f}$ (cf. Eq. 8).

$$k_e = 1.996{,}530{,}8(1 + a' + g'/2 + f') \times 10^{-2}$$

and

$$g_e = 978{,}036.8(1 + g' + \tfrac{1}{3}f')$$

where

$$a' = \frac{6{,}378{,}270 - a_e}{a_e} - \frac{4f'}{3}, \; g' = 0, \; f' = \frac{1}{f} - \frac{1}{297},$$

and A = the atmospheric mass = 0.88×10^{-6}. See Herrick, Baker, and Hilton (39) 1958.

The J_i, coefficients of the zonal harmonics in the Earth's gravitational potential given in Table 4, have recently been derived from the observed motions of artificial satellites: e.g., 1958 Beta 2, 1959 Iota and 1958 Eta (Kozai), 1958 Beta 2 (O'Keefe, Eckels, Squires), 1957 Beta 1 and 1958 Beta 2 (Jacchia), 1957 Beta, 1958 Beta 2 and 1959 Iota (King-Hele) etc. O'Keefe and Eckels (82) found a perturbation in

TABLE 3

Flattening of the Earth

1/f	Author
298.38 ± 0.07	O'Keefe (79) 1958
298.28 ± 0.11	Jacchia (47) 1958
298.32 ± 0.05	Lecar, Sorenson, Eckels (64) 1959
298.24	O'Keefe, Eckels, Squires (81) 1959
298.20 ± 0.03	King-Hele, Merson (56) 1959
298.24 ± 0.02	King-Hele (55) 1960
298.30 ± 0.03	Kozai (43) 1960 (calculated from his J_i by formula (8))
298.24 ± 0.01	Kaula (53) 1961
298.2 ± 0.1	de Vaucouleurs (1961) (30) Weighted mean
298.30 ± 0.05	Adopted[a]

[a] Adopted since it encompasses most of the more recent determinations and is consistent with Kozai's values adopted for the J's.

TABLE 4

Coefficients of Gravitational Harmonics (all values × 10^{-6})

J_2	J_3	J_4	J_5	J_6	Author
1092.8					de Sitter (25) 1924
1092.5 ± 0.7					de Sitter (26) 1927
1094.3 ± 4.4					Jeffreys (49) 1936
1094.08 ± 0.65					de Sitter, Brouwer (27) 1938
1091.8 ± 4.2					Jeffreys (51) 1941
1094.3 ± 2.4					Spencer Jones (98) 1941
1098.21					Clemence (19) 1948
1089.7 ± 2.1					Jeffreys (50) 1948
	−2.9 ± 0.8				O'Keefe, Eckels (82) 1958
1082.1	−2.20 ± .07	−2.15			Kozai (58) 1959
1082.1 ± 0.4		−2.359 ± .002			Lecar, Sorenson, Eckels (64) 1959
1082.53	−2.4 ± 0.3	−1.7	−0.1 ± .1		O'Keefe, Eckels, Squires (81) 1959
	−2.32				Cohen, Anderle (21) 1960)
1082.79 ± 0.15		−1.4 ± 0.2		0.9 ± 0.8	King-Hele (55) 1960
	−2.6		−0.57		Michielsen (71) 1960
1082.190 ± 0.023	−2.29 ± 0.02	−2.13 ± .04	−0.23 ± .02		Kozai (to be published) 1960 (3 sat lites)
1082.28 ± 0.03	−2.29 ± 0.02	−2.12 ± .05	−0.23 ± .02		Hilton recalculation of Kozai valu consistent with Table 6 (43) 19
1082.66	−2.5	−1.7	+0.3	+0.7	Michielsen (72) 1961 (perigee radi determination)
1082.61 ± 0.06	−2.05 ± 0.1	−1.43 ± 0.06	−0.08 ± 0.11	+0.20 ± 0.05	Kaula (53) 1961
1082.28 ± 0.3	−2.3 ± 0.2	−2.12 ± 0.5	−0.2 ± 0.1	+1.0 ± 0.8[a]	Adopted (1961)[b]

[a] J_6 is not necessarily consistent with the other adopted values, as they have been calculated assuming no sixth harmonic.
[b] Adopted from the Hilton revision of Kozai's values, which were based upon observations of three satellites.

TABLE 5

Equatorial Radius of Earth

a_e (meters)	Author	Comments
6,378,301	Clarke (15) 1880	See (36) p. 312
6,378,298 ± 34	Ledersteger (65) 1951	
6,378,228	Hayford Revision (93) 1953	
6,377,879 ± 357	Hirose (45) 1955	
6,378,250 ± 95	"Hough Ellipsoid" (13) 1956	Free Air
6,378,240 ± 100	"Hough Ellipsoid" (13) 1956	Free air with Jeffreys' components
6,378,285 ± 100	"Hough Ellipsoid" (13) 1956	Isostatic
6,378,270 ± 100	Herrick, Baker, Hilton (39) 1958	$1/f = 297$
6,378,145 ± 50	Herrick, Baker, Hilton (39) 1958	Corrected for $1/f = 298.3 \pm 0.1$
6,378,388	Yaplee, Bruton, Craig, Roman (110) 1958	Radar echoes from Moon
6,378,175 ± 20	Yaplee, Bruton, Craig, Roman (110) 1958	Recomputed by C. G. Hilton with revised lur parallax and flattening
6,378,200	Fischer (1959)	World wide areas, recomputed by Cook (22) wi $1/f = 298.2$
6,378,200 ± 30	de Vaucouleurs (30) 1961	Weighted mean adopted
6.378,163 ± 21	Kaula (53) 1961	
6,378,255 ± 35	Yaplee, Bruton, Miller (111) 1960	Radar echoes from moon with $1/f = 298.25 \pm 0.$ and $m_\oplus/m_☾ = 81.375 \pm 0.026$
6,378,150 ± 50	Adopted (1961)	

the eccentricity of 1958 Beta 2 having a period equal to that of the revolution of the line of apsides, which theory predicts if a third order term in the potential with coefficient J_3 is assumed to exist. This indicates that to the third order at least, the Earth is somewhat "pear shaped," i.e. the northern hemisphere has a longer axial semidiameter and a shorter mid-latitudinal semidiameter than does the southern hemisphere (cf. S. W. Carey, "North-South Asymmetry of the Earth's Figure," *Science* **130,** 1960, p. 978; O'Keefe, Eckels and Squires (79), and also King Hele and M son (56)).

3. Selenocentric Constants

3.1. *Lunar Mass.* The ratio of the Moon's mass that of the Earth ($m_☾$) is related to the ratio of t solar and lunar parallaxes through the expression

$$L = \frac{m_☾}{1 + m_☾} \frac{\Pi_\odot}{\sin \Pi_☾} \quad ($$

TABLE 6

Adopted Geocentric Constants (1961)

$_e$ = 6,378,150 (1 ± 11 × 10^{-6}) meters	Adopted from Table 5
$_e$ = 1.996,503,0 (1 ± 11 × 10^{-6}) × 10^{-2} megameters$^{3/2}$ second	Herrick 1958, corrected for new 1/f
$_e$ = 9.780,320 (1 ± 3 × 10^{-6}) meters/sec^2	Herrick, Baker, Hilton (39) 1958
f = 298.30 ± 0.05	Adopted from Table 3
$_2$ = (1082.28 ± 0.3) × 10^{-6}	Adopted from Table 4
$_3$ = (−2.3 ± 0.2) × 10^{-6}	Adopted from Table 4
$_4$ = (−2.12 ± 0.5) × 10^{-6}	Adopted from Table 4
$_5$ = (−0.2 ± 0.1) × 10^{-6}	Adopted from Table 4
$_6$ = (1.0 ± 0.8) × 10^{-6}	Adopted from Table 4

TABLE 7

Determinations of the Moon's Mass (Earth's mass = 1)

, Constant of Lunar Equation from bservation	Adopted Solar Parallax	Reciprocal of Moon's Mass $m^{-1}_{☾}$	Author
	8″.803	81.53	Hinks (44) 1909, Am. Ephm. (5) 1960
6.428,3	8.803	81.530	de Sitter, Brouwer (27) 1938
6.404,9	8.803	81.829	de Sitter, Brouwer (27) 1938
6.406,8	8.803	81.805	de Sitter, Brouwer (27) 1938
6.439,0	8.790	81.271 ± 0.021	Spencer Jones (98) 1941
6.436,7	8.798,4	81.375 ± 0.026	E. Rabe (85) 1949 from Eros
6.442,8	8.790	81.222 ± 0.027	Delano (23) 1950 from right ascensions of Eros
6.443,0	8.790	81.219 ± 0.030	Delano (23) from declinations of Eros
6.438,5 0.001,5	8.798	81.357 ± 0.02	de Vaucouleurs (30) adopted 1961
		81.35 ± 0.050	Adopted (1961)[a]

[a] Rabe's value still appears to be superior.

here L is the coefficient of the "lunar equation," hich is involved in the reduction of geocentric co-dinates to the center of gravity of the Earth-Moon stem (barycenter). Several determinations of the oon's mass (in units of the Earth's mass) were based the close approach of 433 Eros in 1930–31. Assuming e solar and lunar parallaxes to have been found eviously, the method consists in comparing the served positions of Eros when nearest the Earth th an accurate ephemeris, fitting the residuals to a nooth curve that has the periodicity and zero points the lunar equation, and using the curve to improve e adopted value of L. Some results are shown in ble 7. Improved values of the Moon's mass will no ubt be obtained from lunar satellites.

3.2. *Lunar Distance.* Until recently the accepted lues for the Moon's distance and parallax stemmed

TABLE 8

Distance of the Moon

Distance (km)	Author
384,400 ± 4	Christie and Gill (14) 1911; Brown (12) 1919; Lambert (63) 1928; Am. Ephm. (5) 1962.
384,407.6 ± 4.7	O'Keefe and Anderson (80) 1952
384,403 ± 1	O'Keefe 1958
384,402.0 ± 1.2	Yaplee and associates (111) 1960, using equatorial radius of Moon = 1740 ± 1 km
384,400.9 ± 0.7	de Vaucouleurs (30) 1961 using $m_{☾}$ = 81.36 ± 0.02 dynamical parallax = 3422″.524 ± 0.035 for $1/f$ = 300
384,402.0 ± 1.0	de Vaucouleurs weighted mean 1961
384,402 ± 2	Adopted (1961)

TABLE 9

Miscellaneous Selenocentric Constants

f^*	I	Author
0.67 ± 0.03	1° 32′ 00″ ± 14″	I. V. Belkovich (8) 1949, on Crater Mösting A
0.65 ± 0.045	1 32 04 ± 15	A. A. Nefedjev, (75) 1950 on Crater Mösting A
	1 33 50 ± 19	C. B. Watts (105) 1955
0.85 ± 0.03	1 32 49 ± 12	A. A. Yakovkin (109) 1950, from Banachiewiecz 1910–1915 observations
0.60	1 31 22	Weimer, 1954
	1 32 06	Hayn (37) 1907, p. 499, Am. Ephm. (5) 1962.
0.75	1 32 20	Hayn; cf Weimer 1954.

from meridian observations of the Moon's declination made during 1906–1910 from Cape of Good Hope and Greenwich Observatories by Christie and Gill (14). The reduction of their measures, which required an assumption as to the ellipticity of the meridian between the two stations, gave the following results:

$$\text{Moon's parallax} = 57' \, 02''.70$$

"constant of the sine parallax"

$$= 3{,}422''.54 \text{ (i.e. sine } p/\text{sine } 1'')$$

$$\text{distance} = 384{,}400 \text{ km.}$$

These values are still published in the American Ephemeris (1962, p. 485).

Lambert (63) in 1928 reconsidered the reduction of the observations by deriving geometrical lunar parallaxes for values of the ellipticity of the meridian, $1/f$, ranging from 292 to 300, and compared them with equivalent values of the dynamical parallaxes, using as constants[6]

[6] This value of g, though somewhat higher than accepted values, was considered appropriate in view of gravity measurements from a submarine made by Vening Meinesz (1927) on a voyage from Holland to Java, which indicated that gravity at sea slightly exceeds gravity on land.

a_e = 6,378,388 km (International Ellipsoid)

g = 978,052 cm/sec^2

μ^{-1} = 81.53 (Hinks' value from Eros).

Lambert found that for geometrical and dynamical parallax equal to 57′ 02″.72, $1/f$ = 293.5. For the more acceptable flattening of the International Ellipsoid (297) a discrepancy of 0″.16 might be interpreted due to inexact constants used in computing the d namical parallax, or to errors in the deflection of t vertical, the elevation of the geoid above the spher or the observations of Crater Mösting A used in d riving the geometrical parallax. He therefore retain Christie and Gill's value until more observational da from other meridians should become available. Tal

TABLE 10

Reciprocal Masses of the Planets (Sun's Mass = 1)

Planet	m^{-1}	Author	Method
Mercury	6,000,000	Newcomb (76) 1895	
		(Adopted by Am. Ephm. (5))	
		Clemence (16) 1953	
	7,500,000 ± 1,500,000	de Sitter, Brouwer (27) 1938	weighted mean
	6,120,000 ± 43,000	Rabe (85) 1949	Eros
	5,970,000 ± 460,000	Duncombe (31a) 1956	Venus
	6,100,000 ± 50,000	Adopted (1961)	
Venus	408,000	Newcomb (76) 1895	
		(Adopted by Am. Ephm. (5))	
	406,358 ± 723	Fotheringham (34) 1926	Earth, Mars and Mercury
	403,490 ± 2,400	Ross (90) 1916	Mars
	404,700 ± 800	Spencer Jones (97) 1926	Sun
	404,000 ± 1,000	de Sitter (27) 1938	weighted mean
	407,000 ± 500	Morgan, Scott (73) 1939	Sun
	409,300 ± 1,400	Clemence (18) 1943	Mercury
	406,645 ± 208	Rabe (85) 1949	Eros
	407,000 ± 1,000	Adopted (1961)	
Earth + Moon	329,390	Newcomb (76) 1895	
		(Adopted by Am. Ephm. (5))	
	327,900 ± 200	de Sitter (27) 1938	weighted mean
	328,390 ± 103	Witt (106) 1933	Eros
	328,452 ± 43	Rabe (85) 1949	Eros
	328,446 ± 43	E. Rabe (85a) 1954	Eros revised for precession
	328,440 ± 40	de Vaucouleurs (30) 1961	Adopted
	328,450 ± 50	Adopted (1961)	
Mars	3,648,000	Leveau (67) 1890	Vesta
	3,093,500	Newcomb (76) 1895	
		(Adopted by Am. Ephm. (5))	
	3,601,280	Leveau (66) 1907	Vesta
	3,085,000 ± 5,000	de Sitter (27) 1938	weighted mean
	3,110,000 ± 7,700	Rabe (85) 1949	Eros
	3,079,000 ± 5,702	Urey (103) 1952	Deimos
	3,090,000 ± 10,000	Adopted (1961)	
Jupiter	1,050.36	Encke (33) 1938	Vesta
	1,051.42	Hansen (36) 1865	Egeria
	1,047.538	Krüger (60) 1865	Themis
	1,045.63	Leveau (67) 1890	Vesta
	1,047.355 ± 0.065	Newcomb (76) 1895	weighted mean
		(Adopted by Am. Ephm. (5))	
	1,047.34	Newcomb (77) 1895	Polyhymnia
	1,046.04	Leveau (66) 1904	Vesta
	1,047.40 ± 0.03	de Sitter (27) 1938	weighted mean
	1,047.4 ± 0.1	Adopted (1961)	
Saturn	3,501.9	Newcomb (76) 1895	weighted mean
	3,490 ± 5	de Sitter (27) 1938	weighted mean
	3,497.64 ± 0.27	Hertz (41) 1953	
	3,499.7 ± 0.4	Clemence (16) 1953 and (17) 1960	Jupiter perturbations
	3,500 ± 3	Adopted (1961)	

TABLE 10—*Continued*

Planet	m^{-1}	Author	Method
Uranus	22,869	Newcomb (76) 1895 (Adopted by Am. Ephm. (5))	
		Clemence (16) 1953	
	22,750 ± 200	de Sitter (27) 1938	
	22,800 ± 100	Adopted (1961)	
Neptune	19,314	Clemence (16) 1953 (Adopted by Am. Ephm. (5))	
	19,700	Newcomb (76) 1895	
	19,500 ± 200	de Sitter (27) 1938	
	19,500 ± 200	Adopted (1961)	
Pluto	332,488 ± 76,472	Nicholson and Mayall (78) 1931[a]	
	360,000	Clemence (16) 1953	
	307,000	Kuiper (62) 1950	
	350,000 ± 50,000	Adopted (1961)	

[a] Actually given as 1.08 × Earth's mass ± 0.23. In *Pub. A.S.P.*, *43*, no. 74, 1931, the authors give Pluto's mass as $\frac{2}{3}$ the Earth's ass.

compares the older results for the Moon's distance ith those of O'Keefe and Anderson (80) derived from notoelectrically recorded star occultations (1952) nd with the results of radar reflection observations y Yaplee, Bruton, Craig, Roman and Miller (1957–960).

3.3. *Figure of Moon.* The Moon's figure appears to e best approximated by a triaxial ellipsoid with emiaxes a, directed toward the center of the Earth, coincident with the rotational axis, and b, perendicular to a and c. Alexandrov's (4) values derived om Yakovkin's data on the Moon's physical libraon, given below, indicate that it is immaterial whether ne librations are assumed to be forced or free:

	forced libration	free libration	adopted (1961)
semiaxis a	1738.67 ± 0.07	1738.57 ± 0.07	1738.57 ± 0.07
semiaxis b	1738.21 ± 0.07	1738.31 ± 0.07	1738.21 ± 0.07
semiaxis c	1737.58 ± 0.07	1738.58 ± 0.07	1737.58 ± 0.07

Ioments of inertia about the respective axes:

$$= \frac{m}{5}(b^2 + c^2), \qquad B = \frac{m}{5}(c^2 + a^2), \qquad C = \frac{m}{5}(a^2 + b^2) \tag{10}$$

ere derived by Alexandrov as follows:

	forced libration	free libration
$A \times 10^{41}$ gm/km²	88.837 ± 0.024	88.838 ± 0.024
$B \times 10^{41}$ gm/km²	88.856 ± 0.024	88.856 ± 0.024
$C \times 10^{41}$ gm/km²	88.893 ± 0.024	88.893 ± 0.024

Jeffreys (48) finds the following relations:

$$\frac{C - A}{B} = 0.000{,}626{,}6 \pm 0.000{,}002{,}7 \text{ (standard error)}$$

$$\frac{A - B}{C} = 0.000{,}204{,}9 \pm 0.000{,}000{,}9 \text{ for a forced libration}$$

$$= 0.000{,}209{,}8 \pm 0.000{,}002{,}2 \text{ for a free libration.}$$

3.4. *Miscellaneous Selenocentric Constants.* If a function of the moments of inertia about the principal axes is defined by

$$f^* = \left[\frac{C - B}{C - A}\right]\frac{B}{A} \tag{11}$$

its value and that of the inclination of the Moon's equator to the ecliptic can be found from observations of the physical libration. Following in Table 9 are some recent determinations from the Transactions of the I.A.U., 1952 and 1955.

H. Hirose and R. Manabe (45), from observations of occultations of the Pleiades, applying Hayn's corrections for irregularities of the limb, obtained a value for the Moon's semidiameter of

$$932''.80 \pm 0''.07.$$

The value of the ratio of the Moon's diameter to that of the Earth, used by H. M. Nautical Almanac office in occultation predictions, i.e.,

$$k = 0.272{,}495{,}3$$

gives a semidiameter of 1,738,016 ± 14 meters, assuming a_e = 6,378,150 ± 50 meters.

4. Planetocentric Constants

Heliocentric constants are important over a major part of an interplanetary trajectory, but the terminal

phase is influenced by such constants as the mass, diameter, and flattening of the target planet. Planetary distances in astronomical units and geocentric and heliocentric coordinates are generally well established. Planet masses enter into the equations of motion of a vehicle through the perturbative terms. Older values of the masses, found from planetary satellites, the perturbative action of one planet on another, or on a minor planet or satellite, are in the p[rocess] of refinement to meet the need for greater [ac]curacy. In his definitive differential correction of [the] orbit of Eros, Rabe (85), for example, derived [im]provements to the masses of the four inner planets [as] well as to precessional and other constants and cert[ain] elements of the Earth's orbit. Similar improveme[nts] of planetary masses will be derived from observatio[ns]

TABLE 11

Planetary Diameters (Angular Diameters Are Reduced to 1 a.u. Unless Otherwise Stated)

Planet	Linear Diameter (km)	Angular Diameter	Author
Mercury	4,842	6″.68	Leverrier (5) 1843 (Adopted by Am. Ephm.)
	4,670	6.45	Dollfus (88) 1953
	4,650	6.42	Muller (88) 1953
	4,676 ± 31		de Vaucouleurs (31) 1961
	4,660 ± 30		Adopted (1961)
Venus[a]	12,191	16.82	Auwers (5) 1894 (Adopted by Am. Ephm.)
	12,640		W. Rabe (87) 1928
	12,513		Ross (91) 1928
	12,060	16.64	Muller (74) 1948
	12,246		Kuiper (61) 1949
	12,200 ± 10[b]		Menzel, de Vaucouleurs (70) 1960
	12,200 ± 20		Adopted (1961)
Mars	6,743 ± 22	9.268 9.222 (polar)	See (94) 1901
	6,784	9.36	Hartwig (5) 1879 (Adopted by Am. Ephm.)
	6.652 ± 11 6,577 ± 34 (polar)	9.175 ± 0.015	Trumpler (101) 1927
	6,679 ± 42.3	9.48 ± 0.006	Van de Kamp (104) 1928
	6,860 ± 21.7		W. Rabe (87) 1929
	6,834 ± 50.2	9.44 ± 0.07	Reuyl (89) 1941
	6,826	9.29 (polar)	Muller (74) 1948
	6,830		de Vaucouleurs (31)
	6,830 ± 10		Adopted (1961)
Jupiter	142,745 133,236 (polar)	196″.94 183.82	Sampson (5) 1910 (Adopted by Am. Ephm.)
	142,750 ± 100		Adopted (1961)
Saturn	120,798 108,096 (polar)	166.66 149.14	H. Struve (5) 1898 (Adopted by Am. Ephm.)
	121,000 ± 100		Adopted (1961)
Uranus	49,693	68.56	weighted mean, Barnard (5) 1896, See (5) 19[..] Wirtz (5) 1912, (Adopted by Am. Ephm.)
	49,700 ± 100		Adopted (1961)
Neptune	52,999	73.12	Barnard (5) 1902 (Adopted by Am. Ephm.)
	44,600	2.044 at 30.07 a.u.	Kuiper (62) 1949
	50,000 ± 500		Adopted (1961)
Pluto	0.46 × Earth's diameter	0.23 ± 0.01 at 35.56 a.u.	Kuiper (62) 1950
	3,000 ± 1000		Adopted (1961)

[a] Diameter of atmospheric cloud level shell.

[b] Angular diameter up to half-intensity point of atmosphere on measurement of occultation of Regulus by Venus was 17″.00 ± 0[...] and linear diameter 12,321 ± 8 km. They estimate this half-intensity point to be 65 km. above the cloud level (± 10 km) so t[hat] the linear diameter of the cloud shell is approximately 12,200 ± 10 km.

uring close approaches of Icarus (38), Griqua (86), 'riberga (11) and several other minor planets which ave been discovered in recent years.

Table 10 contains the principal determinations of lanetary masses, including the atmospheres and atellites where these are present. The diversity in he values of linear planetary diameters given in Table 1 illustrates the difficulty of obtaining precise measures f the angular diameters (reduced to a distance of 1 .u.) especially in the case of planets with atmospheres. Iore accurate determinations are greatly needed for adar measurements of distance and, particularly, hen landings are to be attempted, since positional lata always refer to a planet's center.

The flattening of a planet, defined as $(a - b)/a$ where a and b are the equatorial and polar radii respectively, and the gravitational harmonics, J_i, become important when orbital motions of nearby satellites are under consideration. They may be found rom dynamical relations, assuming hydrostatic equilibrium and a particular density distribution, from he motion of a planetary satellite, while the flattening nay be measured on photographs directly. Table 12 ontains some estimates of the flattening of Mars, 'able 13a values for the flattening and J_i of Jupiter, Table 13b values for the flattening and J_i of Saturn, and Table 13c some adopted gravitational constants for four planets.

TABLE 12

Flattening of Mars

1/f	Author	Method
92	Struve (99) 1895	dynamical
91.8	Woolard (108) 1944	dynamical
95	Trumpler (101) 1927	photographs
76.9	de Vaucouleurs (29) 1950	weighted mean of optical measures covering 100 years.
77 ± 30	Dollfus (88) 1953	optical in yellow light
67	Muller (88) 1953	optical
59	Camichel (88) 1954	optical
50 ± 50	Adopted (1961)[a]	

[a] Adopted in order to encompass more reliable dynamical eterminations and Trumpler's photographic determination.)'Keefe and Kopal have suggested in conversation that the iscrepancy between optical and dynamical determinations nay be due to the depression of the tropopause in the Northern egions of Mars.

TABLE 13a

Flattening and Gravitational Harmonics of Jupiter

1/f	J_2	J_4	Author and Method
15.0			Sampson (5) 1910, from polar and equatorial radius
15.2	0.014,93	−0.000,405	Sampson (92) 1921, from satellites
	0.014,90	−0.001,0	Brouwer (10a) 1946
15.3	0.014,71	−0.000,336	De Marcus (24) 1959
15.2 ± 0.1	0.014,8 ±0.000,1	−0.000,35 ±0.000,05	Adopted (1961)

TABLE 13b

Flattening and Gravitational Harmonies of Saturn

1/f	J_2	J_4	Author
10.2	1.016,67	−0.000,515	DeMarcus (24) 1959
	0.016,107 ±0.000,007	−0,000,865 ±0.000,026	Kozai (59a) 1957
10.2	0.016,10 ±0.000,01	−0.000,865 ±0.000,026	Adopted 1961

TABLE 13c

Adopted Values of Gravitational Harmonics and Flattening of the Planets

Planet	1/f	J_2	J_4	Comments
Mars	150±50	0.002,011		1/f from Table 12 and J_2 from De Marcus (24) 1959
Jupiter	15.2±0.1	0.014,8 ±0.000,1	−0.000,35 ±0.000,05	Adopted from Table 13a
Saturn	10.2	0.016,10 ±0.000,01	−0.000,865 ±0.000,026	Adopted from Table 13b
Neptune	58.5	0.004,9		De Marcus (24) 1959

5. Atmospheric Constants

Experience with geocentric artificial satellites has already demonstrated the importance of atmospheric constants to astrodynamics. In fact the atmospheric densities at high altitudes have required revision of an order of magnitude over older accepted values in order to agree with satellite drag perturbations. Because the Earth's atmosphere has been discussed in detail in various sources (cf (6) and (7) for example), the following section will treat primarily the atmospheres of Venus and Mars as being of special immediate interest to the space traveler. Atmospheric constants include composition and density at surface and their variation with height; diurnal, seasonal, solar and latitudinal fluctuation in density; and winds. As interplanetary probes make available improved astronomical data, both the quantity and accuracy of atmospheric characteristics will undoubtedly improve.

The variation of density with altitude is best defined in terms of equation (57) from (7), i.e.

$$\ln \rho = \ln \rho_b - \left(1 + \frac{m_0 g_0{}^*}{R^* L_m}\right) \cdot \left\{ \ln \left[1 + \frac{L_m}{(T_m)_b} (H^* - H_b{}^*) \right] \right\} \tag{12}$$

where the geopotential altitude for a spherical planet is

$$H^* = \frac{H g_0}{g_0{}^*(1 + H/R')} . \tag{13}$$

Thus, given values for the base altitudes, H_b^*; the base molecular scale temperature, $(T_m)_b$; the base mean molecular weight, m_b; and the temperature gradient, L_m; from Table 14, an atmospheric density profile can be constructed. The sea level composition of planetary atmospheres is given in Table 15.

The question of a model atmospheric wind pattern is almost meaningless even for the Earth since the atmosphere is subject to forces and stresses at a altitudes that result in complicated flow pattern Hence rather large scale unpredictable variation from any model would be the rule, not the exceptio At this time, if an adoption of a wind model is to b made, the summary work published by J. M. Sok (95) is recommended.

The seasonal, diurnal, solar, latitudinal, etc. var

TABLE 14

Atmospheric Structure Model

Planet	H_b^* (km)	$(T_m)_b$ (°K)	m_b	P_b[a] (Newton/m²)	ρ_b[b] (kg/m³)	Temperature gradient L_m $(H_b < H < H_{max})$ (°C/km)	
Venus[m]	0	340 ± 15	42.4[c]	(2.2 ± 1) × 10⁵[c]	4 ± 3	−5	$0<H<20$[c]
		(580)[k]		(10 to 30 × 10⁵)[k]			
		(535–675)[l]					
		(350)[e]					
$g_o^* = 8.42$ m/sec²	18 ± 3	250 ± 30[c]	42.4[c]	(1.7 ± 2) × 10⁵[c]	3 ± 2	+10	$20<H<30$[c]
	(cloud level)	(285)[k]		(1.1 × 10⁵)[k]		+9[k]	
$R' = 6{,}100{,}000$ m	70 ± 10	240 ± 20[d]	~40	?	?	−1.5	$70<H<80$[d]
	85 ± 10	297 ± 10[e]	~40	0.26 ± 0.13[e]	(4 ± 1) × 10⁻⁶	+3	$80<H<100$[e]
Earth:	0	288.16	28.966	1.01 × 10⁵	1.2250	−6.5	$0<H<11$
$g_o^* = 9.80665$	11	216.66	28.966	2.28 × 10⁴	3.6480 × 10⁻¹	0	$11<H<25$
m/sec²							
$R' = 6{,}356{,}766$ m	25	216.66	28.966	2.52 × 10³	4.0639 × 10⁻²	+3.0	$25 < H < 47$
	47	282.66	28.966	1.25 × 10²	1.5535 × 10⁻³	0	$47 < H < 53$
	53	282.66	28.966	6.14 × 10¹	7.5791 × 10⁻⁴	−4.5	$53 < H < 79$
	79	165.66	28.97	1.23	2.520 × 10⁻⁵	0	$79 < H < 90$
	90	165.66	28.97	1.35 × 10⁻¹	2.846 × 10⁻⁶	+4.0	$90 < H < 105$
	105	225.66	28.86	9.66 × 10⁻²	1.542 × 10⁻⁷	+20	$105 < H < 160$
	160	1.325.66	28.04	4.10 × 10⁻³	1.123 × 10⁻⁹	+10	$160 < H < 170$
	170	1.425.66	27.15	3.14 × 10⁻³	7.932 × 10⁻¹⁰	+5	$170 < H < 200$
	200	1.575.66	26.32[g]	1.65 × 10⁻³	3.673 × 10⁻¹⁰	+3.5	$200 < H < 70$
Mars	0	290 ± 15[e]	28.3[c]	(8.7 ± 0.4) × 10³	(0.99 ± 0.05) × 10⁻¹	−4	$0 < H < 30$
		(218 ± 76°)[l]	(44 to 92)[i]				
$g_o^* = 3.91$ m/sec²	30 ± 10[h]	180 ± 20[e]	~28	(1.4 ± 0.2) × 10³	(2.7 ± 1) × 10⁻²	−1	$30 < H < 90$
$R' = 3{,}400{,}000$ m	90 ± 20[h]	140 ± 10[e]	~28	7.2 ± 2	(1.5 ± 1) × 10⁻⁴	+1.5	$90 < H < 110$
		(130)[h]				0	$200 < H < 350$
	200 ± 50[j]	300 ± 50[j]	~28	~1	~1 × 10⁻⁵		
	350 ± 50[j]	300 ± 50[j]	28 to 14[j]	~1 × 10⁻²	~1 × 10⁻⁷	~50	$350 < H < 550$

[a] 10⁵ Newtons/meter² = 1 bar. The surface pressure is determined in the case of Mars by molecular light scattering. De Vaucouleurs (28) 1960 finds $P_{bo} = (8.5 \pm 0.4) \times 10^3\ nt/m^2$, while Urey (102) 1959 modifies Dollfus' 1957 value to $8.9 \times 10^3\ nt/m^2$.

[b] $\rho_b = P_b m_b/(T_m)_b R^*$.

[c] Urey (102) 1959; approximate values for the uncertainties added by the authors; pressure of $1.7 \times 10^5\ nt/m^2$ at 18 ± 3 km abov Venus' surface was taken as basis for computation of "sea level" values.

[d] Pettit and Nicholson (83) 1955.

[e] Differential refraction studies of photoelectric observations of the occultation of Regulus by Venus, Menzel and de Vaucouleur (70), under the assumptions that $g = 8.6$ m/sec², 90% CO_2, 9% N_2, and 1% A+ ··· . We assume 65 km above clouds or about 85 k above surface.

[f] ARDC Model Atmosphere (6) 1959.

[g] From an altitude of 180 km on, the molecular weight is obtained from $m = 27.106 - 7.935{,}697{,}10 \tan^{-1} \dfrac{(H - 180)}{140}$.

[h] Kopal (57) 1960.

[i] Kiess, Karrer and Kiess (54) 1960. Such a molecular weight would result in a two to four times more dense atmosphere tha has been quoted here.

[j] Estimated from tables given by Yanow (112) 1961 for $\rho_{bo} = 10^{-1}$ kg/m³.

[k] Barrett (7a) 1961 found on the basis of radio astronomical observations of Venus, assuming 75% CO_2, 22–25% N_2 and 0 to 3 H_2O.

[l] Mayer, McCullough, and Sloanaker (68a) 1958, find a surface temperature for Venus of 620 ± 110°K 40 days before inferi conjunction, and 560 ± 73°K at conjunction. In 1960 (68b) they found radio surface temperatures of 535 ± 65°K 16 days aft conjunction and 675 ± 85°K, 39 days after conjunction.

[m] A recent paper by Carl Sagan (*Science 133*, 849–858) gives an excellent summary of our present knowledge of Venus, including i atmosphere. Sagan seems to favor a hot dry surface with an atmospheric pressure of about 4 newtons/m² and a temperature of abo 600°K, assuming a temperature gradient of −10 (0 < H < 35 km) and a tropospheric height of about 23 km above the planet's surfac

TABLE 15

Atmospheric Composition at Surface (% by Volume)

Planet	N_2	O_2	CO_2	H_2O	A^{40}	CH_4	NH_3	He	Other
enus[a]	15[b] (22 to 25)[g]	<10[e]	80[e] (75)[g]	variable $\frac{1}{10}$ to $\frac{1}{20}$ of Earth's (0 to 3)[g]	—	detected[b]	detected[b]	—	CO, N_2O, C_2H_4, C_2H_6
lopted (1961)	10 ± 10	5 ± 2	80 ± 15	assumed saturated					
arth lopted 961)	78.084[e]	20.946[e]	0.033[e]	variable	0.934[e]	2×10^{-4e}	—	5×10^{-4e}	Ne, Kr, Xe, H_2, N_2O
ars[d]	>80[e]	<0.03[e]	1.9[e] 0.33[i]	0.029[e]	<20[e]	detected[e]	detected[e]	—	CO^+, SO_2, O_3, N_2O, CH_4, C_2H_4, C_2H_2
	98.5[f]	<0.14[g]	0.25[f]		1.2[f]				
	98[h]		0.66[h]		1.2[h]				
	93.8[e]	<0.1[e]	2.2[e]		4.0[e]				
lopted (1961)	96 ± 2	0.06 ± .04	2 ± 1	0.03 ± 0.01	1.5 ± 1				

[a] above cloud layer.
[b] Kuiper (61) 1949.
[e] Urey (102) 1959.
[d] Kiess, Karrer and Kiess (54) 1960, propose that the atmosphere of ars is composed of Nitrogen oxides; principally N_2O and N_2O_4.
[e] de Vaucouleurs (28) 1960.
[f] de Vaucouleurs (29) 1950.
[g] Barrett (7a) 1961.
[h] Kopal (57) 1960.
[i] Hess (42) 1958.

ion in the atmospheric densities, like the winds, are ot only quite variable but as yet not completely efinitized. Consequently we can only recommend to e reader the research in the field by L. G. Jacchia 6) and by H. A. Martin and W. Priester (68) and opose no model for adoption at this time.

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Rigid Body Attitude Stability and Natural Frequencies in a Circular Orbit[1]

D. B. DeBra and R. H. Delp[2]

Introduction

The equations of attitude motion of a small rigid body in orbit have been recorded frequently in the literature; e.g., [1], [2]. Subject to several restrictions and assumptions they can be reduced to a set of three simultaneous linear, ordinary differential equations with constant coefficients. The conditions are that: (1) the only forces on the body are representable by an attractive inverse-square central-force field about an unaccelerated point; (2) the body is small enough so that attitude motions have no significant effect on the orbital motion of the center of mass; (3) the orbit is circular; (4) the attitude deviations from the equilibrium position are small. The equations given in [2] are based on these assumptions. They describe the attitude behavior of a rigid body subject to gravity gradient torques in a circular orbit of angular speed ω_0.

Stability

With the coordinates of Figure 1 the equilibrium positions correspond to the body principal axes aligned with the coordinates. The principal moments of inertia about these axes are I_1, I_2, I_3 respectively.

From [2] the equations for small motions about this equilibrium position are:

$$\begin{gathered} I_1\ddot{\theta}_1 + (I_3 - I_2)\omega_0^2\,\theta_1 - \omega_0\dot{\theta}_2(I_1 + I_2 - I_3) = 0 \\ \omega_0\dot{\theta}_1(I_1 + I_2 - I_3) + I_2\ddot{\theta}_2 + (I_3 - I_1)\,\omega_0^2\theta_2 = -3\omega_0^2\,(I_3 - I_1)\theta_2 \\ I_3\ddot{\theta}_3 = -3\,\omega_0^2(I_2 - I_1)\theta_3. \end{gathered} \tag{1}$$

Taking the Laplace transform; replacing the frequency variable s by $p = (s/\omega_0)^2$; introducing moment of inertia parameters by setting:

$$(I_3 - I_2)/I_1 = k_1, \qquad (I_1 - I_3)/I_2 = k_2 \quad \text{and} \quad (I_2 - I_1)/I_3 = k_3 \tag{2}$$

and simplifying,[3] the equation can be written in matrix form as:

$$\begin{bmatrix} (p + k_1) & -\sqrt{p}(1 - k_1) & 0 \\ \sqrt{p}\,(1 + k_2) & (p - 4k_2) & 0 \\ 0 & 0 & (p + 3k_3) \end{bmatrix} \begin{bmatrix} \theta_1\,(p) \\ \theta_2\,(p) \\ \theta_3\,(p) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

Since the equations are homogenous, the behavior obtained from the characteristic equation which is jus the determinant of the square matrix in (3):

$$[p^2 + (1 - 3k_2 - k_1k_2)\,p - 4k_1k_2]\,(p + 3k_3) = 0 \tag{4}$$

This equation is a cubic in $p = (s/\omega_0)^2$. For th equilibrium to be stable it is necessary that all the root of (4) be real and negative. The conditions this place on the coefficients of (4) are therefore conditions on th shape of the body for stability to exist. Together wit the physical limitations on the k's, a chart can b prepared describing the necessary shape of a body fc stability to exist (Figure 2). These conditions are:

Condition I

For a principal moment of inertia where x_2 and are the coordinates of an element of mass dm, I_1

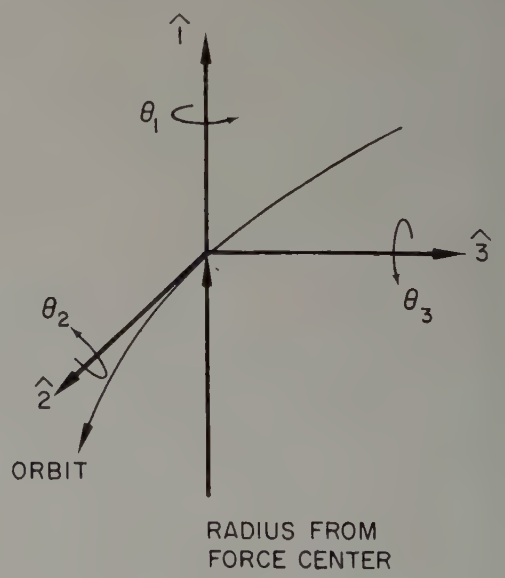

FIG. 1. Orthogonal set of unit vectors defining the local o bit coordinates and the small angles which define body attitu relative to these coordinates.

[1] Received October 1960.

[2] Lockheed Aircraft Corporation, Sunnyvale, California.

[3] When this study was made in 1958, the moment of inertia ratios I_1/I_3 and I_2/I_3 were used [3]. In March 1960 R. Smelt introduced the notation of (2) in his course "Dynamics of Space Vehicles" at Stanford University. Smelt's notation considerably simplifies the analysis.

$\int (x_2^2 + x_3^2)\, dm$. Then

$$|k_1| = \left|\frac{I_3 - I_2}{I_1}\right| = \left|\frac{\int (x_2^2 - x_3^2)\, dm}{\int (x_2^2 + x_3^2)\, dm}\right| < 1$$

and similarly $-1 < k_2 < 1$, and $-1 < k_3 < 1$ for any physical body.

Condition II

From (4) the factored root requires $k_3 > 0$.

Condition III

Again from (4) the quadratic formula must give two negative real roots for p; hence:

(A) $\qquad (1 - 3k_2 - k_1k_2) > 0$

(B) $\qquad (1 - 3k_2 - k_1k_2)^2 + 16\, k_1k_2 > 0$

and

(C) $\qquad k_1k_2 < 0.$

Because any two of the k's describe the shape of the body, one of the k's may be eliminated. From (2) $k_1 + k_2 + k_3 + k_1\, k_2\, k_3 = 0$ so that:

$$k_1 = -(k_2 + k_3)/(1 + k_2k_3)$$
$$k_2 = -(k_3 + k_1)/(1 + k_3k_1) \qquad (5)$$

and

$$k_3 = -(k_1 + k_2)/(1 + k_1k_2).$$

In view of the complexity of Condition III, k_3 will be eliminated. Since it follows from Condition I that $1 + k_1k_2 > 0$, Condition II becomes $k_2 + k_1 < 0$. The stable regions in a k_1k_2 plot are therefore bounded by parts of the following curves:

Condition I

$k_1 = -1$ and $k_2 = -1,$

Condition II

$k_1 = -k_2,$

Condition III

(B) $k_2 = -\,[(7k_1 - 3) + 4\sqrt{3k_1\,(k_1 - 1)}]/(k_1 + 3)^2$

(C) $k_1 = 0$ and $k_2 = 0.$

Condition III (A) and parts of each of the other conditions are weaker and do not contribute to the boundary.

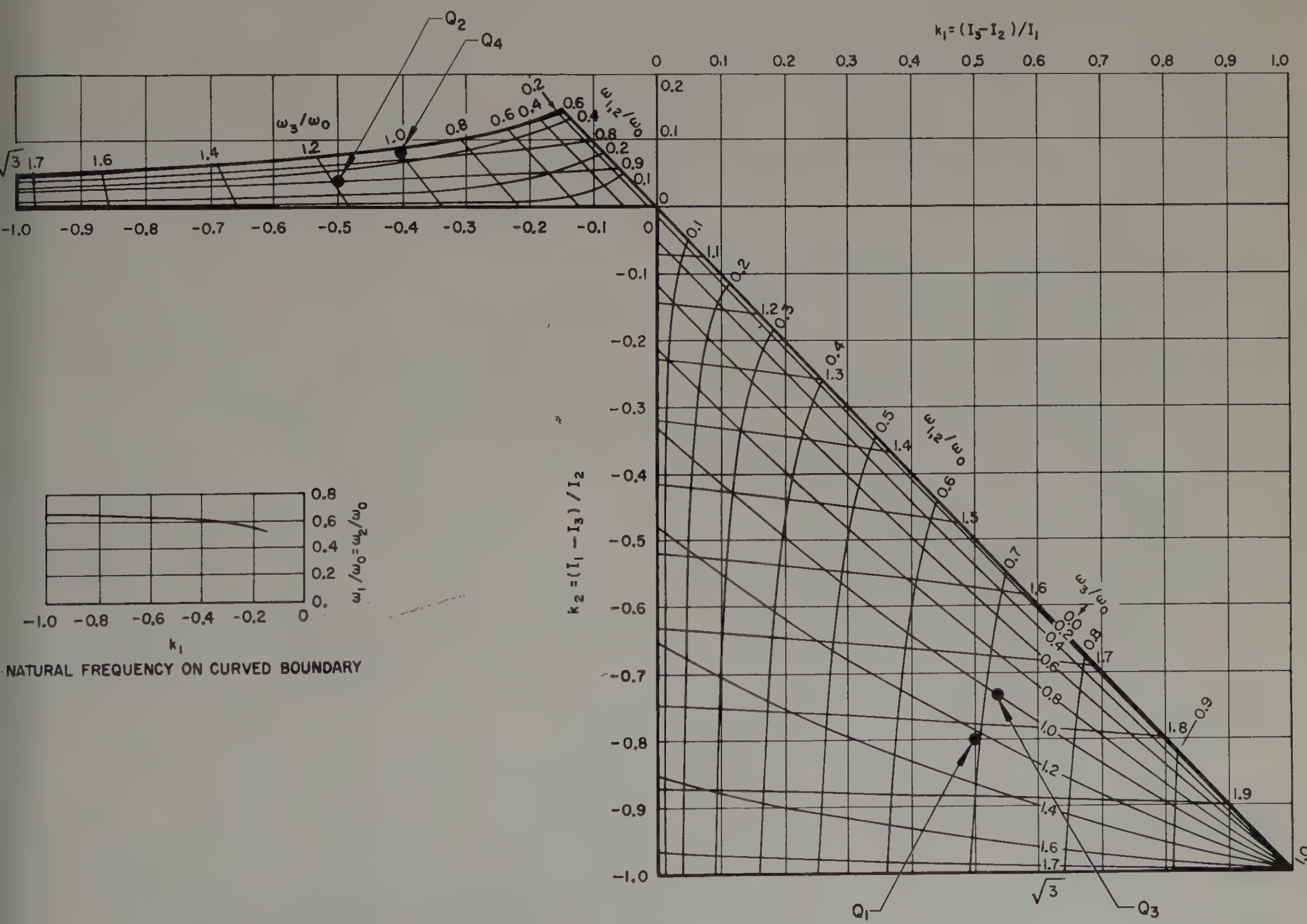

FIG. 2. Attitude stability and natural frequencies of a rigid body in a circular orbit plotted versus k_1 and k_2.

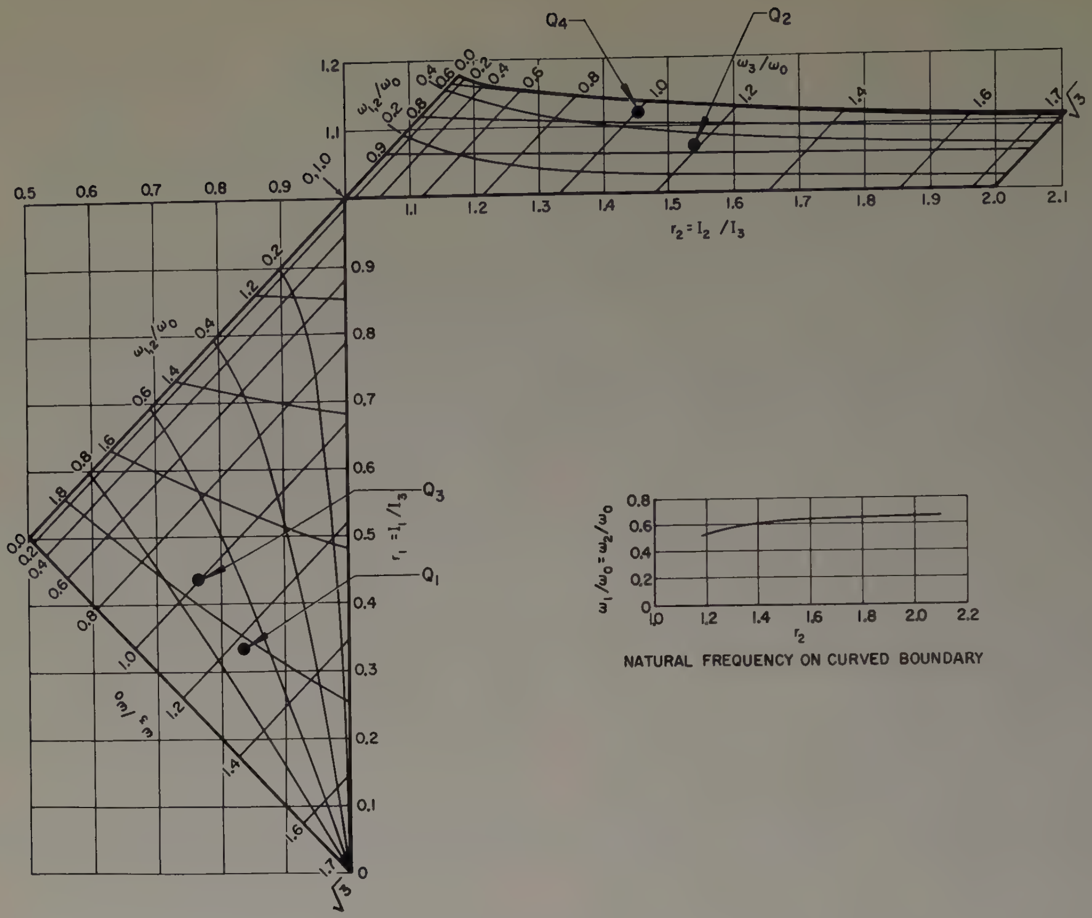

FIG. 3. Attitude stability and natural frequencies of a rigid body in a circular orbit plotted versus r_1 and r_2.

The resultant regions of stability are shown in Figure 2 and Figure 3.

Natural Frequencies

For each point in the stable region there are three natural frequencies associated with the natural motions of the body attitude. The motion about the $\hat{1}$ and $\hat{2}$ axes is coupled, so both the natural frequencies from the quadratic part of (4) are present in the θ_1 and θ_2 motions. These frequencies are called ω_1 and ω_2. Since they may be incommensurable, the resultant Lissajous figure swept out by the tip of $\hat{3}$ may not be closed; i.e., it can be a conditionally periodic motion. The θ_3 motion occurs about the $\hat{3}$ axis at frequency ω_3.

The natural frequencies can be obtained from (4) and expressed in terms of any two of the k's using (5). If $p_1 = -(\omega_1/\omega_0)^2$, $p_2 = -(\omega_2/\omega_0)^2$ and $p_3 = -(\omega_3/\omega_0)^2$ and k_1 and k_2 are given:

$$p_{1,2} = (-1 + 3k_2 + k_1 k_2 \pm \cdot\sqrt{1(-3k_2 - k_1 k_2)^2 + 16 k_1 k_2})/2 \tag{6}$$

and

$$p_3 = 3(k_1 + k_2)/(1 + k_1 k_2).$$

The explicit expressions in terms of the other two combinations of the k's can be obtained by direct substitution using (5) again. As an example Table I has been prepared for two sets of values of k_1 and k_2. The corre sponding points are shown in Figure 2 as point Q and Q_2.

Though the analysis is simpler using the k parameters the ratios $r_1 = I_1/I_3$ and $r_2 = I_2/I_3$ give a simple visualization of the physical body. It is therefore usefu to have the relations between the k parameters anc the r parameters. From (2), (7) gives the moment o inertia ratios as functions of the k's taken in pairs For example the r_1 and r_2 corresponding to points Q and Q_2 can be computed from (7) and are included i Table I.

$$r_1 = \frac{1 + k_2}{1 + k_1 k_2} = \frac{1 - k_3}{1 + k_1} = \frac{1 + k_2 k_3}{1 - k_2}$$
$$r_2 = \frac{1 - k_1}{1 + k_1 k_2} = \frac{1 + k_1 k_3}{1 + k_1} = \frac{1 + k_3}{1 - k_2} \tag{7}$$

The inverse functions are:

$$k_1 = (1 - r_2)/r_1$$
$$k_2 = (r_1 - 1)/r_2 \tag{8}$$

and

$$k_3 = r_2 - r_1.$$

In order to avoid resonances or to achieve certai dynamic behavior, constraints may be placed on th natural frequencies. It is then desired to determi

TABLE I

k_1	k_2	Point	k_3	$\frac{\omega_1}{\omega_0}$	$\frac{\omega_2}{\omega_0}$	$\frac{\omega_3}{\omega_0}$	r_1	r_2
0.5	−0.8	Q_1	0.5	0.695	1.82	1.225	0.333	0.833
−0.5	0.04	Q_2	0.469	0.3165	0.928	1.187	1.06	1.53

whether these frequencies are realizable and to find the body shape as described by the k's or the r's.

Only two of the natural frequencies are independent, however, since two parameters describe the shape of the body. Because of the coupling of the motion between the $\hat{1}$ and $\hat{2}$ axes ω_1 and ω_2 cannot be distinguished from each other. There are therefore only two ways the frequencies may be given, viz. ω_1 and ω_2, or ω_3 and either ω_1 or ω_2. Then from (4) if ω_1 and ω_2 are given (again $p_1 = -(\omega_1/\omega_0)^2$ etc.).

$$k_1 = -3p_1 p_2/[p_1 p_2 + 4(1 + p_1 + p_2)]$$

$$k_2 = [p_1 p_2 + 4(1 + p_1 + p_2)]/12 \tag{9}$$

$$k_3 = \frac{36p_1 p_2 - [p_1 p_2 + 4(1 + p_1 p_2)]^2}{3(4 - p_1 p_2)[p_1 p_2 + 4(1 + p_1 + p_2)]}$$

or given ω_1 and ω_3

$$k_1 = (A \pm \sqrt{A^2 - B})/6\,(p_1 + 4)$$

where

$$A = p_3(p_1 + 2p_1 + 4) - 9p_1$$

$$B = 36p_1(p_1 + 4)(1 + p_1 - p_3)$$

$$k_2 = C \pm \sqrt{C^2 - D} \tag{10}$$

where

$$C = (4p_3 - 9p_1 - 2\,p_1p_3 - p_1^2p_3)/6(p_1 + 3p_1p_3 - 4)$$

$$D = p_1(p_1 + 1)/(p_1 + 3p_1p_3 - 4)$$

$$k_3 = -\,p_3/3.$$

An example is worked in Table II for a given ω_1 and ω_3. Two possible configurations are obtained from (10) and are shown on Figure 2 as points Q_3 and Q_4.

For even slight eccentricity there is parametric and forced excitation at ω_0. The points Q_3 and Q_4 and any other point where a natural frequency is equal or near orbit frequency are excited near resonance and would be particularly poor choices.

Under some circumstances the body configuration may be a secondary consideration to the frequencies. Given two natural frequencies it may be desired to obtain the third frequency directly rather than by successively solving (9) or (10) and then (6). The expression for the third frequency given ω_1 and ω_3 is:

$$p_2 = (-2B \pm \sqrt{4B^2 - 16AC})/A \tag{11}$$

where

$$A = p_1^2(1 + p_3) + 4p_1\,(2 + p_3) + 16$$

$$B = p_1^2(2 + p_3) + p_1 + 4(2 - p_3)$$

$$C = p_1^2 + p_1(2 - p_3) + (1 - p_3)$$

TABLE II

$\frac{\omega_1}{\omega_0}$	$\frac{\omega_3}{\omega_0}$	Point	k_1	k_2	k_3	$\frac{\omega_2}{\omega_0}$	r_1	r_2
0.707	1.0	Q_3	0.527	−0.733	0.333	1.76	0.433	0.768
		Q_4	−0.407	0.0856	0.333	0.524	1.124	1.458

and given ω_1 and ω_2 it is

$$p_3 = \frac{[p_1 p_2 + 4(1 + p_1 + p_2)]^2 - 36p_1 p_2}{(4 - p_1 p_2)[p_1 p_2 + 4(1 + p_1 + p_2)]}. \tag{12}$$

Though the preceding formulae are all useful for calculating precise results, they are slow for preliminary design studies. The loci of natural frequencies have therefore been plotted on the stable regions of Figure 2 and 3. (These loci are for the normalized frequencies ω_1/ω_0 etc. not the p's.) Any pair of parameters (except ω_3 and k_3) may be used to enter the curves and locate a point. For example given ω_3 and r_1 the designer can immediately read off the associated ω_1, ω_2 and r_2 from Figure 3.

Conclusion

In order that a small rigid body in a circular orbit may have an orientation which is a position of stable equilibrium in all three axes, it is necessary that it have a mass distribution which is one of two types. The first type corresponds to the stable region in the fourth quadrant of the k_1k_2 plot of Figure 2, and is the type usually referred to in the literature. It is characterized by $I_1 < I_2 < I_3$. The second type corresponds to a stable region in the second quadrant of the $k_1\,k_2$ plot. In this second region the vehicles are characterized by $I_3 < I_1 < I_2$. Vehicles of the first type have a mass distribution similar to a board flying edgewise along the orbit with its long axis vertical. The second type is like a board flying broadside along the orbit with the long axis normal to the orbit plane. The second type is stable because the destabilizing gravity gradient effect about the $\hat{2}$ axis is weaker than the gyroscopic stabilization due to angular velocity about the $\hat{3}$ axis.

For the assumptions made at the beginning of this note not more than a total of two shape parameters and/or frequencies may be chosen independently. Any pair (except ω_3 and k_3) completely determine the other shape parameters and frequencies.

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Analysis of Error Progression in Terminal Guidance for Lunar Landing[1]

P. J. deFries[2]

Abstract

A lunar descent scheme with n periods of engine ON-OFF is investigated. Each period may have a different thrust level but it is constant for the length of the period. The engines are ignited by altitude and cut off by velocity signals. Five groups of errors are considered: ignition altitude (Δs), velocity change (Δu), thrust level (ΔF), lunar gravitation (Δg), and initial approach velocity (Δw_0). Equations are derived that allow the computation of the influence of any given error occurring during any one of the n periods on the final velocity at touchdown.

It is concluded that it is principally advantageous to break up the descent into ON-OFF periods. Also it is inferred, as far as guidance is concerned, that the problem of descent rests with the very last period of braking.

The upper periods can always be handled with constant thrust and rather crude instrumentation whereas the instrumentation for the last period depends upon the required softness of the landing which also determines whether or not variable thrust is necessary.

LIST OF SYMBOLS

M	Mass
$\dot{M}$	Mass per unit time expelled by rocket motor
M_0	Initial mass
t	Time
g	Gravitational acceleration
g_0	Gravitational acceleration of moon at surface
g^*	Gravitational acceleration of moon at end of braking period
y	Momentary displacement of vehicle (vertical)
$\dot{y}$	Momentary velocity of vehicle (vertical)
F	Thrust of rocket motor
ln	Natural logarithm
s	Displacement at beginning of braking period
w	Velocity at end of free fall period
w_0	Initial velocity of approach
c	Velocity of exhaust gases of rocket motor
u	Velocity change due to rocket motor
v	Velocity at end of a braking period
R	Radius of moon
K	Gravitational parameter of moon
h	Displacement at end of braking period
i	Subscript designating any one ON-OFF period
n	Subscript designating last of ON-OFF periods
m	Subscript defined in Appendix II

Introduction

A lunar landing is an example of an exclusively rocket-powered descent of a space vehicle without aerodynamic influence either to assist in the landing operation or to hinder in dangerous aerothermodynamic heating. If the vehicle is to touch down with zero velocity, then cutoff of the engine must coincide wit zero velocity and zero altitude. If the engine is to b ignited only once, then the altitude at which ignitio must occur is determined by the approach velocit This, in turn, requires coincidence of a velocity, a altitude and the ignition signal of the engine. Meetin these demands of a total of six parameters is a seve requirement upon any guidance scheme, particularl so if the available thrust is nonvariable. This is obviou without an involved mathematical treatment. It i not so obvious, however, that a multistep approac relaxes the severity of the scheme to such an exten that fairly simple guidance instrumentation become feasible. There are several important properties of th Multi-Step-Scheme from which to determine its use fulness for instrument carrier landings and manne landings. Of particular importance are for exampl the ease and economy with which willful lateral ma neuvers can be performed, (location guidance), th degree of error coupling between lateral and vertica guidance, the degree of interference of the rocket flam with the guidance sensors and the sensitivity of th scheme to the inaccuracies of the guidance sensor This report compiles the results of an investigation int the progression of errors of a multi-step termina guidance scheme based on the following assumptions:

(1) The thrust is nonvariable.
(2) The engine is ignited by an altitude senso
(3) The engine is cutoff by an inertial velocit sensor, i.e., after a velocity change has bee effected.

Guidance errors are considered which result from th following:

(1) The altitude, s_1, at which the engine is ignited is in error by $\pm\Delta s_1$.
(2) The velocity change, u_1, which cuts off th engine, is in error by $\pm\Delta u_1$.
(3) The thrust, F_1, is uncertain to with $\pm\Delta F_1$
(4) The initial velocity of approach, w_0, is un certain to within $\pm\Delta w_0$.

The equation of motion for a braking period canno be handled easily because the gravitational componen is a nonlinear function of altitude and the mass of th vehicle is a function of time. If the gravitation is as sumed to be constant the equation becomes linear an can be treated explicitly. This approach is taken in th analysis. The error introduced thereby is accounte for by consideration of the gravitational acceleratio g, of the celestial body (luna) as an "independen

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variable", with the uncertainty, Δg. This then is a fifth error.

The effect of these five errors on the final velocity, v_1, expresses itself in Δv_1 and is quite severe. It places very stringent requirements on the permissible magnitude of the initial five errors.

Earlier work on the subject, (1), (2), indicated that an immediate relief can be gained by allowing a small free fall period between cutoff and touchdown.

If a certain impact velocity, w_1, is allowed, the effect of the aforementioned five errors on w_1 is decisively less than on v_1. The individual errors are reduced by the mere insertion of this free fall period; this is shown mathematically in Appendix II. Of course, a price must be paid. This is the increase of velocity from v_1 to w_1.

One might go one step further and reignite the engine after the free fall period in order to eliminate or drastically reduce the velocity, w_1. This would permit a low touchdown velocity to be linked to the advantage of an error reduction afforded by the introduction of the free fall period. There is, however, no change in the progression of errors. v_2 is just as strongly affected by the initial errors as w_1. See Appendix II. Another reduction of initial errors could be gained by addition of a second free fall period, resulting in a final velocity, w_2. This would again reduce all errors accumulated up to the second free fall period. Therefore, we reason that if this ON-OFF sequence were continued for a large number of times all errors could be drastically reduced if not practically eliminated. The question to be answered, then, is how the errors progress through the chain of ON-OFF periods. In other words, if there are n ON-OFF periods, it is necessary to determine what effect an error introduced during the ith preceding period would have on the nth period. The equations of motion used for investigating this error progression are derived in Appendix I and are listed as equations (12) through (20).

The present investigation considers a constant thrust level during each period but allows for different thrust levels from period to period. The lunar gravitation is approximated during the powered phases only, not during the free fall. The error incurred in approximation of the lunar gravitation is accounted for by its introduction as an independent error in the error analysis.

Source of Errors

During each period four errors are introduced. Two stem from wrong measurements and two from wrong assumptions. Measurements are taken of the altitude, s, and the velocity change, u. The errors are Δs and Δu. Assumptions are made in thrust, F, and in lunar gravitational acceleration, g. The errors are ΔF and Δg. All periods are alike except for the very first one which has an additional, fifth, error in the initial velocity of approach, w_0. This error is Δw_0.

The velocity at the end of the nth period, w_n, then is a function of:

(1) n altitudes plus their errors $(s_1 + \Delta s_1, s_2 + \Delta s_2, \cdots, s_n + \Delta s_n)$

(2) n velocity changes plus their errors $(u_1 + \Delta u_1, u_2 + \Delta u_2, \cdots, u_n + \Delta u_n)$

(3) n thrust values plus their errors $(F_1 + \Delta F_1, F_2 + \Delta F_2, \cdots, F_n + \Delta F_n)$

(4) n gravity values plus their errors $(g_1 + \Delta g_1, g_2 + \Delta g_2, \cdots, g_n + \Delta g_n)$

(5) one initial velocity plus error $(w_0 + \Delta w_0)$.

For the sake of abbreviated writing we shall use:

$(s + \Delta s)$ meaning $(s_1 + \Delta s_1, s_2 + \Delta s_2, \cdots, s_n + \Delta s_n)$

$(u + \Delta u)$ meaning $(u_1 + \Delta u_1, u_2 + \Delta u_2, \cdots, u_n + \Delta u_n)$

$(F + \Delta F)$ meaning $(F_1 + \Delta F_1, F_2 + \Delta F_2, \cdots, F_n + \Delta F_n)$

$(g + \Delta g)$ meaning $(g_1 + \Delta g_1, g_2 + \Delta g_2, \cdots, g_n + \Delta g_n)$.

Developing w_n in a Taylor series with first order terms only, as shown in equation (21) of Appendix II, gives:

$$w_n(s + \Delta s, u + \Delta u, F + \Delta F, g + \Delta g, w_0 + \Delta w_0) = w_n(s, u, F, g, w_0) + (\Delta s)A + (\Delta u)C + (\Delta F)D + (\Delta g)E + (\Delta w_0)B.$$

Equations (22) thru (26) of Appendix II show what the five Δ-terms stand for namely:

$$(\Delta s)A = (\Delta s_1)A_{1n} + (\Delta s_2)A_{2n} + \cdots + (\Delta s_n)A_{nn} = \sum_{i=1}^{n-1} \Delta s_i A_{in} + (\Delta s_n)A_{nn}$$

and analogous for the other coefficients except for equation (26)

$$(\Delta w_0)B = (\Delta w_0)B_{1n}.$$

In equation (32) the Taylor series for w_n thus becomes:

$$w_n(s + \Delta s, u + \Delta u, F + \Delta F, g + \Delta g, w_0 + \Delta w_0) = w_n(s, u, F, g, w_0) + \sum_{i=1}^{n-1} (\Delta s_i)A_{in} + \sum_{i=1}^{n-1} (\Delta u_i)C_{in} + \sum_{i=1}^{n-1} (\Delta F_i)D_{in} + \sum_{i=1}^{n-1} (\Delta g_i)E_{in} + (\Delta w_0)B_{1n} + (\Delta s_n)A_{nn} + (\Delta u_n)C_{nn} + (\Delta F_n)D_{nn} + (\Delta g_n)E_{nn}.$$

The Δ-terms in equations (32) are easily interpreted as to their physical meaning. The last four terms, with only the index n, represent the errors introduced during the last period itself. The term with coefficient B is

representative of the error present in the last period because of an error in the initial velocity of approach. Finally, the five terms with the $\sum$ represent the errors that are carried down into the last period from the errors introduced during all the preceding periods.

The coefficients A, C, D, E, B, with their respective indices can be resolved to:

$$A_{in} = A_{ii}\frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right)\Phi$$

$$C_{in} = C_{ii}\frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right)\Phi$$

$$D_{in} = D_{ii}\frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right)\Phi$$

$$E_{in} = E_{ii}\frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right)\Phi$$

$$B_{1n} = B_{11}\frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right)\Phi$$

$$\Phi = w_i \prod \left(\frac{u_{i+m}}{w_{i+m-1}}\right), \qquad m = 1, 2, \cdots (n - i - 1).$$

(See equations 69 through 74 of Appendix II.)

All of these coefficients are of the same mathematical structure. In other words, all errors progress alike through the ON-OFF periods.

In the equations (69) through (73) a choice has been made with respect to the constant value of the gravitational acceleration of the moon during the braking phases. It is advantageous to choose the constant g to be the acceleration existing at the moment of cut-off of the rocket motor. The reasons are explained in Appendix II. The braking phases are thus always calculated with too large a gravitational acceleration meaning that the ambiguous $\pm$ sign with Δg is resolved into a definite minus. By use of equations (69) through (73), the Taylor series of equation (32) becomes as shown in equation (75), which follows:

$$\begin{aligned} w_n(s + \Delta s, u + \Delta u, F + \Delta F, g + \Delta g, w_0 + \Delta w_0) \\ = w_n(s, u, F, g, w_0) + \frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right) \\ \left[\sum \Delta s_i A_{ii}\Phi + \sum \Delta u_i C_{ii}\Phi + \sum \Delta F_i D_{ii}\Phi \right. \\ \left. + \sum \Delta g_i E_{ii}\Phi + \Delta w_0 B_{11}\Phi\right] + (\Delta s_n)A_{nn} \\ + (\Delta u_n)C_{nn} + (\Delta F_n)D_{nn} + (\Delta g_n)E_{nn} \end{aligned}$$

$$\Phi = w_i \prod \left(1 - \frac{u_{i+m}}{w_{i+m-1}}\right); \qquad m = 1, 2, \cdots, (n - i - 1)$$

$$\Phi = w_i \qquad\qquad n - i - 1 = 0.$$

Equation (74), above, defines the Φ-terms. The nine error terms of equation (75) are examined critically in the following section.

Equation (75) readily allows the computation of the so-called partials or error coefficients of a given descent program. These partials are weighing factors which relate the deviation of the final touchdow velocity from the nominal to the deviation of the fiv variables from nominal. As an indication of the valuab information that they supply; note the following thre partials for the firing altitudes in the descent prograr shown in Figure 1. The lower the period the mor critical the partials become. There is nearly a two orde of magnitude difference between them.

$$\frac{\partial w_3}{\partial s_1} = 0.059 \cdot 10^{-3}\left[\frac{m/s}{m}\right]$$

$$\frac{\partial w_3}{\partial s_2} = -3.5 \cdot 10^{-3}\left[\frac{m/s}{m}\right]$$

$$\frac{\partial w_3}{\partial s_1} = 110 \cdot 10^{-3}\left[\frac{m/s}{m}\right]$$

Critical Evaluation

In a purely mathematical sense all errors excep the ones occurring during the last period could b eliminated if $u_n/w_{n-1} = 1$. (See equation 75). How ever, this can be fulfilled only if w_{n-1} is known accu rately, i.e., by measurement of w_{n-1}. This renders th condition of this equation trivial in a physical sens since, if w_{n-1} is known at a certain altitude, previou errors lose significance.

However, a guidance scheme that measures w_{n-} with its associated altitude while neglecting the meas urement accuracies in the upper periods might b adopted. In that case, the question remains whethe the errors accumulated in the earlier braking phase would allow close enough approach to the surface s that measurements for the last period could be carrie out safely and so that the remaining altitude would b neither too high nor too low for the actual landing t be made. This physical statement interpreted i mathematical terms of equation (75) considers wha conditions minimize the individual terms in the bracke

Consider the series of products Φ, common to a terms, and the individual coefficients A_{ii}, C_{ii}, D_{ii} E_{ii}, B_{11}. Φ could be very effective in reducing th

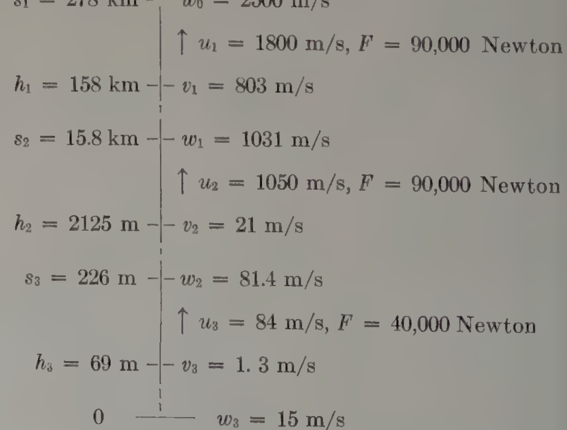

FIG. 1. Descent program. This is an arbitrary program whic is not optimized or selected with any special conditions i mind. It is for illustration purposes only.

ıultitude of errors. It becomes small if the same ondition as that mentioned above

$$\frac{u_n}{w_{n-1}} = 1$$

; met for all or any one of the other periods, that is if

$$\frac{u_{i+m}}{w_{i+m-1}} = 1\,.$$

3ut even if this were only approached rather roughly he effect would be very pronounced since the product eries of relatively small factors decreases rapidly. The effectiveness of Φ decreases with the lower periods ince the product series becomes shorter. Many ON-)FF periods would, therefore, be advantageous for educing errors. For the period immediately preceding he last one the effectiveness of Φ is lost and the co-fficients A_{ii}, C_{ii}, D_{ii}, E_{ii} must be considered for ninimizing the errors introduced during this period. This latter point will not be discussed in more detail ince all the necessary conclusion can be drawn from he subsequent discussion of the coefficients in the ast ON-OFF period by simply changing the indices.

The errors introduced during the last period are epresented by the last four terms in equation (75).)ne may dismiss the very last one, $(\Delta g_n)E_{nn}$, since he last period will be so close to the surface that the unar gravitational acceleration may safely be assumed constant. This means $\Delta g_n = 0$.

The first coefficient, A_{nn} (equation 94), presents a lear cut situation. It cannot go to zero. This means hat the altitude error when the rocket motor is fired or the last time is there to stay. A_{nn} can only be mini-nized by a large w_n. This, however, is severely limited y the demanded softness of the landing. Hence, the reatest efforts must be made in accurately measuring ltitude before firing the engine for the last time.

Next is C_{nn} (equation 95). It could go to zero if $_n = 0$. This is expressed in equation (101) in terms f the velocity change, namely:

$$u_n = w_{n-1} + \frac{g_n M_{0n}}{F_n}\left(1 - e^{-(u_n)/c}\right) = w_{n-1} + g_n t_n$$

large w_n would also be beneficial.

The last term is D_{nn} (equation 96). Strictly mathe-natically speaking it could go to zero, but for all ractical reasons it cannot since the condition for $D_{nn} = 0$ is only met by a ratio of $u_n/c = 0$ (103, Ap-endix II). A compromise u_n/c will have to be chosen. large thrust to weight ratio would generally be ıvorable although at the same time one would prefer ow exhaust velocity (notice c^2) which means low pecific impulse. The dependency of D_{nn} on c is rela-ively weak, however.

This evaluation does not yet account for the measure-ient of w_{n-1} which in turn introduces an error (Δw_{n-1}). he coefficient with (Δw_{n-1}) would be B_{nn} and is analogous to B_{11}.

$$B_{nn} = \frac{w_{n-1} - u_n}{w_n}$$

For $w_{n-1} = u_n$ the coefficient B_{nn} is zero. In other words the error introduced by measuring the velocity w_{n-1} before the last braking starts is not critical.

Conclusions

From the standpoint of minimizing the effects of inaccuracies of measuring devices the following philosophy of a lunar terminal guidance with constant thrust engines deserves attention. The vertical descent should be broken up into a number of ON-OFF periods. The guidance effort should be concentrated upon highly accurate instrumentation for the last period, minimizing the errors by proper choice of the velocity change and fully exploiting the allowances in the landing velocity. For all other periods, the errors can be held down with rather simple instrumentation by proper choice of the velocity changes, the number of ON-OFF periods and the spacing of the ON-phases to the OFF-phases.

The guidance for the last period demands highly accurate instrumentation. In particular it requires an altimeter for determining the altitude of firing, a velocity meter for measuring the velocity before firing and a velocity meter for measuring the change of velocity once the rocket motor burns.

This latter velocity meter is certain to be of the inertial type and is readily available. The other velocity meter and altimeter must be selected with due consideration for the possible dust stirred up from the surface by the rocket flame. Since the last period will certainly be initiated below 1000 meters altitude a doppler type velocity meter and a radio altimeter would be familiar electronic instruments which would have none of the problems associated with high altitude altimeters and doppler equipment.

If the allowed touchdown velocity is high enough (in the order of 15 m/s) the velocity measurement before firing can be omitted altogether, thus requiring only an altimeter and the integrating gyro.

For the upper periods a crude altimeter would suffice for initiating the braking phases. A horizon sensor which is required for establishing the local vertical possibly could supply the necessary altitude signals. The velocity data for the upper periods may be taken from calculated trajectory data and integrating gyros thus requiring little instrumentation.

The thrust level in the upper periods is constant during each period but there are no paramount requirements for any specific thrust level from the standpoint of guidance errors, although a high thrust to weight ratio is always advantageous (compare equation 96). There is also freedom to change the thrust level from period to period. The foregoing analysis allowed for a different thrust level in each period. This particular

point will probably have little meaning for landing instrument carriers but a manned vehicle is certain to be restricted in the permissible deceleration and a reduction in thrust from higher to lower periods for the purpose of keeping the thrust to weight ratio within narrow limits may be attractive.

The vertical guidance would impose little restrictions on the lateral maneuvers during the upper periods. The lateral maneuvers would enjoy a large degree of freedom within the limits of fuel economy.

The unavoidable errors of the last period must be kept within bounds by the length of the final free fall before touchdown; in other words, the velocity of landing, w_n, is the principal means of holding the errors down. For any one of the guidance errors the velocity dispersion at touchdown is always inversely proportional to w_n.

The most important are the altitude error and the thrust error. The error in velocity change and the one introduced by measuring w_{n-1} are of second order significance. Theoretically they can be eliminated altogether whereas the first two cannot.

If the demands for the softness of the landing exceed the minimum landing velocity necessary for keeping the errors small (compared to the vertical displacement and the vertical velocity of the vehicle), the constant thrust engine cannot be maintained. The alternate solution would be to aim with the scheme described so far (having constant thrust in each braking phase) at a touchdown at around 200 meters above the surface and switching then to a second smaller rocket motor with variable thrust; the thrust would be controlled by velocity and altitude continuously. In this case the instrumentation is especially concerned with the ionization effects of the flame and the dust problem. Approximately a 10:1 range in thrust variability will be required. However, it is doubtful whether it is worthwhile to attempt an extremely soft landing (landing velocity in the order of 1 m/s or less in contrast to 10 to 20 m/s for an ordinary soft landing) without a firm knowledge of the lunar surface. Measurements of the altimeter and the velocity meter would have to be so accurate that only very specific ground properties would allow us to obtain these accuracies. Present day knowledge of the lunar surface definitely does not give us the required information as to the degree of roughness, dustlayer coverage, flatness, solidity, etc.

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Appendix I

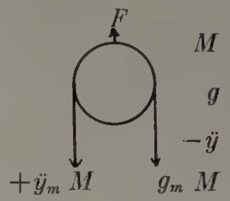
Equations of Motion

Assumptions:

(1) The celestial body to which the vehicle descends has no atmosphere (luna).

(2) The engine is first ignited at an altitude which is small compared to the radius of the celestial body (luna).

Because of assumption (2) the surface under the vehicle is considered infinitely large and the motion in vertical (y) and horizontal direction may be treated independently. Only the vertical motion will be dealt with.



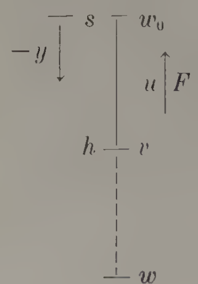
$$M = M(t) = M_0 - \dot{M}(t) \quad (1)$$

$$g = g(y) \quad (2)$$

$$-\ddot{y} = g - \frac{F}{M_0 - \dot{M}t}, \quad (3)$$

g is a function of altitude y. This makes it hard to work with equation (3). We will assume that g is constant during the braking phase and account for this inaccuracy by considering g as a "variable" in the error analysis. The assumed error, Δg, will be determined by the difference between the actual g and the constant g value chosen for the braking phase.

Vertical velocity, $\dot{y}$, and altitude, y, as a function of time, t:



$$\dot{y} = w_0 + gt - c \ln \frac{M_0}{M_0 - \dot{M}t} \quad (4)$$

$$y = s - w_0 t - g\frac{t^2}{2} + c\left[t + \left(t - \frac{M_0 c}{F}\right) \ln \frac{M_0}{M_0 - \dot{M}t}\right] \quad (5)$$

$$u = c \ln \frac{M_0}{M_0 - \dot{M}t}. \quad (6)$$

Vertical velocity, $\dot{y}$, and altitude, y, as a function of the velocity change, u, effected by the braking:

$$= w_0 + \frac{gM_0 c}{F}(1 - e^{-u/c}) - u \quad \text{for} \quad v < \dot{y} < w_0 \tag{7}$$

$$= s - w_0 \frac{M_0 c}{F}(1 - e^{-u/c}) - \frac{g}{2}\frac{M_0^2 c^2}{F^2}(1 - e^{-u/c})^2 + \frac{M_0 c^2}{F}\left[(1 - e^{-u/c}) - \frac{u}{c} e^{-u/c}\right]. \tag{8}$$

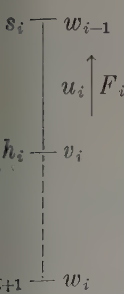
or the free fall following the braking phase the gravitional acceleration is *not* approximated by a constant.

$$-\dot{y}\, d\dot{y} = g\, dy = g_0 \frac{R^2}{y^2} dy \tag{9}$$

$$\dot{y}^2 = 2K\left[\frac{1}{y} - \frac{1}{h}\right] + v^2 \quad \text{for} \quad v < \dot{y} < w \tag{10}$$

$$K = g_0 R^2. \tag{11}$$

ummarizing the essential equations that express the ertical motion in terms of the velocity change, u, and troducing the general index i one finds for one brakg period plus one subsequent free fall:

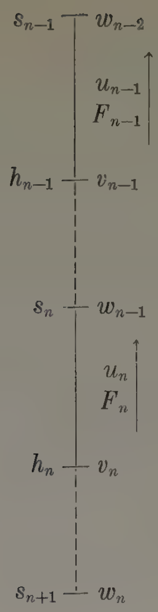


$$v_i = w_{i-1} + g_i \frac{M_{0i} c}{F_i}(1 - e^{-u_i/c}) - u_i \tag{12}$$

$$h_i = s_i - w_{i-1} \frac{M_{0i} c}{F_i}(1 - e^{-u_i/c}) - \frac{g_i}{2}\frac{M_{0i}^2 c^2}{F_i^2}(1 - e^{-u_i/c})^2 + \frac{M_{0i} c^2}{F_i}\left[(1 - e^{-u_i/c}) - \frac{u_i}{c} e^{-u_i/c}\right] \tag{13}$$

$$w_i^2 = 2K\left[\frac{1}{s_{i+1}} - \frac{1}{h_i}\right] + v_i^2 . \tag{14}$$

d the same set of equations for the very last one of periods, n, and the next to last period, $n - 1$,



$$v_{n-1} = w_{n-2} + g_{n-1} \frac{M_{0(n-1)} c}{F_{n-1}} \cdot (1 - e^{-u_{n-1}/c}) - u_{n-1} \tag{15}$$

$$h_{n-1} = s_{n-1} - w_{n-2} \frac{M_{0(n-1)} c}{F_{n-1}}(1 - e^{-u_{n-1}/c}) - \frac{g_{n-1}}{2}\frac{M_{0(n-1)}^2 c^2}{F_{n-1}^2}(1 - e^{u_{n-1}/c})^2 + \frac{M_{0(n-1)} c^2}{F_{n-1}}\left[(1 - e^{-u_{n-1}/c}) - \frac{u_{n-1}}{c} e^{-u_{n-1}/c}\right] \tag{16}$$

$$w_{n-1}^2 = 2K\left[\frac{1}{s_n} - \frac{1}{h_{n-1}}\right] + v_{n-1}^2 \tag{17}$$

$$v_n = w_{n-1} + g_n \frac{M_{0n} c}{F_n}(1 - e^{-u_n/c}) - u_n \tag{18}$$

$$h_n = s_n - w_{n-1} \frac{M_{0n} c}{F_n}(1 - e^{-u_n/c}) - \frac{g_n}{2}\frac{M_{0n}^2 c^2}{F_n^2}(1 - e^{-u_n/c})^2 + \frac{M_{0n} c^2}{F_n}\left[(1 - e^{-u_n/c}) - \frac{u_n}{c} e^{-u_n/c}\right] \tag{19}$$

$$w_n^2 = 2K\left[\frac{1}{s_{n+1}} - \frac{1}{h_n}\right] + v_n^2. \tag{20}$$

Appendix II

Error Progression

Abbreviations:

$s + \Delta s = s_1 + \Delta s_1, \quad s_2 + \Delta s_2, \cdots, s_n + \Delta s_n$

$u + \Delta u = u_1 + \Delta u_1, \quad u_2 + \Delta u_2, \cdots, u_n + \Delta u_n$

$F + \Delta F = F_1 + \Delta F_1, \quad F_2 + \Delta F_2, \cdots, F_n + \Delta F_n$

$g + \Delta g = g_1 + g_1, \quad g_2 + \Delta g_2, \cdots, g_n + \Delta g_n$

Taylor series for w_n with first order terms only:

$$w_n(s + \Delta s\,,\, u + \Delta u,\, F + \Delta F,\, g + \Delta g,\, w_0 + \Delta w_0) \\ = w_n(s, u, F, g, w_0) + (\Delta s)A + (\Delta u)C \\ + (\Delta F)D + (\Delta g)E + (\Delta w_0)B. \quad (21)$$

The last five terms in equation (21) are:

$$(\Delta s)A = (\Delta s_1)A_{1n} + (\Delta s_2)A_{2n} + \cdots \\ + (\Delta s_n)A_{nn} = \left[\sum_{i=1}^{n-1} (\Delta s_i)A_{in}\right] + (\Delta s_n)A_{nn} \quad (22)$$

$$(\Delta u)C = (\Delta u_1)C_{1n} + (\Delta u_2)C_{2n} + \cdots \\ + (\Delta u_n)C_{nn} = \left[\sum_{i=1}^{n-1} (\Delta u_i)C_{in}\right] + (\Delta u_n)C_{nn} \quad (23)$$

$$(\Delta F)D = (\Delta F_1)D_{1n} + (\Delta F_2)D_{2n} + \cdots \\ + (\Delta F_n)D_{nn} = \left[\sum_{i=1}^{n-1} (\Delta F_i)D_{in}\right] + (\Delta F_n)D_{nn} \quad (24)$$

$$(\Delta g)E = (\Delta g_1)E_{1n} + (\Delta g_2)E_{2n} + \cdots \\ + (\Delta g_n)E_{nn} = \left[\sum_{i=1}^{n-1} (\Delta g_i)E_{in}\right] + (\Delta g_n)E_{nn} \quad (25)$$

$$(\Delta w_0)B = (\Delta w_0)B_{1n}. \quad (26)$$

The capital letter symbols are the error coefficients, also referred to as "partials".

$$A_{in} = \frac{\partial w_n}{\partial s_i} \qquad A_{nn} = \frac{\partial w_n}{\partial s_n} \quad (27)$$

$$C_{in} = \frac{\partial w_n}{\partial u_i} \qquad C_{nn} = \frac{\partial w_n}{\partial u_n} \quad (28)$$

$$D_{in} = \frac{\partial w_n}{\partial F_i} \qquad D_{nn} = \frac{\partial w_n}{\partial F_n} \quad (29)$$

$$E_{in} = \frac{\partial w_n}{\partial g_i} \qquad E_{nn} = \frac{\partial w_n}{\partial g_n} \quad (30)$$

$$B_{1n} = \frac{\partial w_n}{\partial w_0} \quad (31)$$

Thus the series of equation (21) becomes:

$$w_n(s + \Delta s,\, u + \Delta u,\, F + \Delta F,\, g + \Delta g,\, w_0 + \Delta w_0) \\ = w_n(s, u, F, g, w_0) \\ + \sum_{i=1}^{n-1} (\Delta s_i)A_{in} + \sum_{i=1}^{n-1} (\Delta u_i)C_{in} + \sum_{i=1}^{n-1} (\Delta F_i)D_{in} \\ + \sum_{i=1}^{n-1} (\Delta g_i)E_{in} \\ + (\Delta w_0)B_{1n} + (\Delta s_n)A_{nn} + (\Delta u_n)C_{nn} \\ + (\Delta F_n)D_{nn} + (\Delta g_n)E_{nn}. \quad (32)$$

To obtain the error coefficients differentiate (15) through (20)

$$A_{in} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial s_i} + \frac{K}{w_n h_n^2}\frac{\partial h_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial s_i} \qquad i \neq n \quad (33)$$

$$C_{in} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial u_i} + \frac{K}{w_n h_n^2}\frac{\partial h_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial u_i} \qquad i \neq n \quad (34)$$

$$D_{in} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial F_i} + \frac{K}{w_n h_n^2}\frac{\partial h_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial F_i} \qquad i \neq n \quad (35)$$

$$E_{in} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial g_i} + \frac{K}{w_n h_n^2}\frac{\partial h_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial g_i} \qquad i \neq n \quad (36)$$

$$B_{1n} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial w_0} + \frac{K}{w_n h_n^2}\frac{\partial h_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial w_0} \quad (37)$$

$$\frac{\partial v_n}{\partial s_i} = \frac{\partial v_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial s_i} = \frac{\partial w_{n-1}}{\partial s_i} \quad (38)$$

$$\frac{\partial v_n}{\partial u_i} = \frac{\partial v_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial u_i} = \frac{\partial w_{n-1}}{\partial u_i} \quad (39)$$

$$\frac{\partial v_n}{\partial F_i} = \frac{\partial v_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial F_i} = \frac{\partial w_{n-1}}{\partial F_i} \quad (40)$$

$$\frac{\partial v_n}{\partial g_i} = \frac{\partial v_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial g_i} = \frac{\partial w_{n-1}}{\partial g_i} \quad (41)$$

$$\frac{\partial v_n}{\partial w_0} = \frac{\partial v_n}{\partial w_{n-1}}\frac{\partial w_{n-1}}{\partial w_0} = \frac{\partial w_{n-1}}{\partial w_0} \quad (42)$$

$$\frac{K}{w_n h_n^2} = \frac{g_n^*}{w_n} \quad (43)$$

(38) through (42) into (33) through (37) and observing (43)

$$A_{in} = \frac{\partial w_{n-1}}{\partial s_i}\left[\frac{v_n}{w_n} + \frac{g_n^*}{w_n}\frac{\partial h_n}{\partial w_{n-1}}\right] \\ C_{in} = \frac{\partial w_{n-1}}{\partial u_i}\left[\frac{v_n}{w_n} + \frac{g_n^*}{w_n}\frac{\partial h_n}{\partial w_{n-1}}\right] \\ D_{in} = \frac{\partial w_{n-1}}{\partial F_i}\left[\frac{v_n}{w_n} + \frac{g_n^*}{w_n}\frac{\partial h_n}{\partial w_{n-1}}\right] \\ E_{in} = \frac{\partial w_{n-1}}{\partial g_i}\left[\frac{v_n}{w_n} + \frac{g_n^*}{w_n}\frac{\partial h_n}{\partial w_{n-1}}\right] \\ B_{1n} = \frac{\partial w_{n-1}}{\partial w_0}\left[\frac{v_n}{w_n} + \frac{g_n^*}{w_n}\frac{\partial h_n}{\partial w_{n-1}}\right] \quad (44)$$

$$\frac{\partial h_n}{\partial w_{n-1}} = -\frac{M_{0n}c}{F_n}(1 - e^{u_n/c}) = -\frac{v_n + u_n - w_{n-1}}{g_n} \quad (45)$$

(45) into (44)

$$A_{in} = \frac{\partial w_{n-1}}{\partial s_i}\frac{1}{w_n}\left[\frac{g_n^*}{g_n}(w_{n-1} - u_n) + v_n\left(1 - \frac{g_n^*}{g_n}\right)\right] \\ C_{in} = \frac{\partial w_{n-1}}{\partial u_i}\frac{1}{w_n}\left[\frac{g_n^*}{g_n}(w_{n-1} - u_n) + v_n\left(1 - \frac{g_n^*}{g_n}\right)\right] \\ D_{in} = \frac{\partial w_{n-1}}{\partial F_i}\frac{1}{w_n}\left[\frac{g_n^*}{g_n}(w_{n-1} - u_n) + v_n\left(1 - \frac{g_n^*}{g_n}\right)\right] \\ E_{in} = \frac{\partial w_{n-1}}{\partial g_i}\frac{1}{w_n}\left[\frac{g_n^*}{g_n}(w_{n-1} - u_n) + v_n\left(1 - \frac{g_n^*}{g_n}\right)\right] \\ B_{1n} = \frac{\partial w_{n-1}}{\partial w_0}\frac{1}{w_n}\left[\frac{g_n^*}{g_n}(w_{n-1} - u_n) + v_n\left(1 - \frac{g_n^*}{g_n}\right)\right]. \quad (46)$$

Equations (46) provide the arguments for choosing he value of g_n. It is obviously advantageous to select

$$g_n = g_n^* \tag{47}$$

espectively

$$g_i = g_i^*. \tag{48}$$

Equations (46) then reduce to the following set of quations for the error coefficients:

$$A_{in} = \frac{\partial w_{n-1}}{\partial s_i} \frac{w_{n-1}}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \tag{49}$$

$$C_{in} = \frac{\partial w_{n-1}}{\partial u_i} \frac{w_{n-1}}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \tag{50}$$

$$D_{in} = \frac{\partial w_{n-1}}{\partial F_i} \frac{w_{n-1}}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \tag{51}$$

$$E_{in} = \frac{\partial w_{n-1}}{\partial g_i} \frac{w_{n-1}}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \tag{52}$$

$$B_{1n} = \frac{\partial w_{n-1}}{\partial w_0} \frac{w_{n-1}}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right). \tag{53}$$

Thus ∂w_n has been reduced to a function of ∂w_{n-1}. n the same way ∂w_{n-1} can be reduced to a function of ∂w_{n-2} and so on until the ith period itself is reached. n other words, from:

$$A_{in} = \frac{\partial w_n}{\partial s_i} = \frac{\partial w_{n-1}}{\partial s_i} \frac{w_{n-1}}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right), \tag{54}$$

ve infer

$$A_{i(n-1)} = \frac{\partial w_{n-1}}{\partial s_i} = \frac{\partial w_{n-2}}{\partial s_i} \frac{w_{n-2}}{w_{n-1}} \left(1 - \frac{u_{n-1}}{w_{n-2}}\right), \tag{55}$$

r in general connotation

$$A_{i(n-k)} = \frac{\partial w_{n-k}}{\partial s_i} = \frac{\partial w_{(n-k-1)}}{\partial s_i} \frac{w_{(n-k-1)}}{w_{(n-k)}} \left(1 - \frac{u_{(n-k)}}{w_{(n-k-1)}}\right). \tag{56}$$

This is also true for $C_{i(n-k)}$, $D_{i(n-k)}$ and $E_{i(n-k)}$. This s true for all integers k except for:

$$n - k = i, \tag{57}$$

$- k = i$ is a special case requiring special attention. For the time being we stipulate as "standard coeffiients":

$$A_{ii} = \frac{\partial w_i}{\partial s_i} \tag{58}$$

$$C_{ii} = \frac{\partial w_i}{\partial u_i} \tag{59}$$

$$D_{ii} = \frac{\partial w_i}{\partial F_i} \tag{60}$$

$$E_{ii} = \frac{\partial w_i}{\partial g_i} \tag{61}$$

$$B_{11} = \frac{\partial w_1}{\partial w_0}. \tag{62}$$

Notice that the last equation does not have the index i. For the sake of ease of the mathematical description of what is to follow it is expedient to rewrite the equations for the coefficients using the index i instead of n. In other words the rest of the text will be more easily explained by looking forward from the ith period to the nth period instead of going backwards from the last, the nth, period to the ith period. In this connotation then the coefficients for the periods following the ith period are (taking only one coefficient as an example):

$$A_{i(i+1)} = A_{ii} \frac{w_i}{w_{i+1}} \left(1 - \frac{u_{i+1}}{w_i}\right) \tag{63}$$

$$A_{i(i+2)} = A_{i(i+1)} \frac{w_{i+1}}{w_{i+2}} \left(1 - \frac{u_{i+2}}{w_{i+1}}\right). \tag{64}$$

In general connotation this is:

$$A_{i(i+m)} = A_{i(i+m-1)} \frac{w_{i+m-1}}{w_{i+m}} \left(1 - \frac{u_{i+m}}{w_{i+m-1}}\right); \quad m = 1, 2 \cdots (n - i). \tag{65}$$

This is the equivalent to (56). Combining (63) through (65):

$$A_{i(i+m)} = A_{ii} \frac{w_i}{w_{i+1}} \left(1 - \frac{u_{i+1}}{w_i}\right) \frac{w_{i+1}}{w_{i+2}} \left(1 - \frac{u_{i+2}}{w_{i+1}}\right) \cdots \frac{w_{(i+m-1)}}{w_{(i+m)}} \left(1 - \frac{u_{(i+m)}}{w_{(i+m-1)}}\right) \tag{66}$$

$$A_{i(i+m)} = A_{ii} \frac{w_i}{w_{i+m}} \left(1 - \frac{u_{i+1}}{w_i}\right)\left(1 - \frac{u_{i+2}}{w_{i+1}}\right) \cdots \left(1 - \frac{u_{i+m}}{w_{i+m-1}}\right) \quad m = 1, 2, \cdots (n - i). \tag{67}$$

And for the nth period finally evolves:

$$A_{in} = A_{ii} \frac{w_i}{w_n} \left(1 - \frac{u_{i+1}}{w_i}\right)\left(1 - \frac{u_{i+2}}{w_{i+1}}\right) \cdots \left(1 - \frac{u_n}{w_{n-1}}\right). \tag{68}$$

Resultingly the error coefficients in equation (32) become:

$$A_{in} = A_{ii} \frac{1}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \Phi \tag{69}$$

$$C_{in} = C_{ii} \frac{1}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \Phi \tag{70}$$

$$D_{in} = D_{ii} \frac{1}{w_n} \left(1 - \frac{w_n}{w_{n-1}}\right) \Phi \tag{71}$$

$$E_{in} = E_{ii} \frac{1}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \Phi \tag{72}$$

$$B_{1n} = B_{11} \frac{1}{w_n} \left(1 - \frac{u_n}{w_{n-1}}\right) \Phi. \tag{73}$$

The operator Φ being:

$$\boxed{\begin{aligned} &\Phi = w_i \prod \left(1 - \frac{u_{i+m}}{w_{i+m-1}}\right) \\ &\qquad\qquad m = 1, 2, \cdots, (n - i - 1) \\ &\Phi = w_i \qquad\quad \text{for } n - i - 1 = 0. \end{aligned}} \tag{74}$$

Enter with (69) through (73) into (32) again and obtain the final form of the series of equation (21) with which the analysis was started:

$$\boxed{\begin{aligned} &w_n(s + \Delta s, u + \Delta u, F + \Delta F, g + \Delta g, w_0 + \Delta w_0) = w_n(s, u, F, g, w_0) \\ &+ \frac{1}{w_n}\left(1 - \frac{u_n}{w_{n-1}}\right)\left[\sum_{i=1}^{n-1} \Delta s_i\, A_{ii}\,\Phi + \sum_{i=1}^{n-1} \Delta u_i\, C_{ii}\,\Phi + \sum_{i=1}^{n-1} \Delta F_i\, D_{ii}\,\Phi + \sum_{i=1}^{n-1} \Delta g_i\, E_{ii}\,\Phi + \Delta w_0\, B_{11}\,\Phi\right] \\ &\qquad + (\Delta s_n)\, A_{nn} + (\Delta u_n)\, C_{nn} + (\Delta F_n)\, D_{nn} + (\Delta g_n)\, E_{nn} \end{aligned}} \tag{75}$$

The last four terms in equation (75) contain the error coefficients of the very last period only. These are:

$$A_{nn} = \frac{K}{w_n\, h_n^2} \tag{76}$$

$$C_{nn} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial u_n} + \frac{K}{w_n\, h_n^2}\frac{\partial h_n}{\partial u_n} \tag{77}$$

$$D_{nn} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial F_n} + \frac{K}{w_n\, h_n^2}\frac{\partial h_n}{\partial F_n} \tag{78}$$

$$E_{nn} = \frac{v_n}{w_n}\frac{\partial v_n}{\partial g_n} + \frac{K}{w_n\, h_n^2}\frac{\partial h_n}{\partial g_n} \tag{79}$$

$$B_{11} = \frac{v_1}{w_1}\frac{\partial v_1}{\partial w_0} + \frac{K}{w_1\, h_1^2}\frac{\partial h_1}{\partial w_0}. \tag{80}$$

Equations (76) thru (80) can be resolved as follows:

$$\begin{aligned} \frac{\partial v_n}{\partial u_n} &= -(v_n + u_n - w_{n-1}) + \frac{G_n\, M_{0n}\, c}{F_n} \\ &\qquad - 1 = \frac{g_n\, M_{0n}}{F_n}\, e^{-u_n/c} - 1 \end{aligned} \tag{81}$$

$$\frac{\partial v_n}{\partial F_n} = -\frac{1}{F_n}(v_n + u_n - w_{n-1}) = -\frac{g_n\, t_n}{F_n} \tag{82}$$

$$\frac{\partial v_n}{\partial g_n} = \frac{1}{g_n}(v_n + u_n - w_{n-1}) = t_n \tag{83}$$

$$\frac{\partial v_1}{\partial w_5} = 1 \tag{84}$$

$$\frac{\partial h_n}{\partial u_n} = -\frac{\partial v_n}{\partial u_n}\frac{v_n}{g_n} - \frac{v_n}{g_n} = -v_n\frac{M_{0n}}{F_n}\, e^{-u_n/c} \tag{85}$$

$$\begin{aligned} \frac{\partial h_n}{\partial F_n} &= \frac{\partial v_n}{\partial F_n}(w_{n-1} - u_n - v_n)\frac{1}{2g_n} \\ &\qquad + \frac{s_n - h_n}{F_n} = \frac{1}{F_n}\left(s_n - h_n + g_n\frac{t_n^2}{2}\right) \end{aligned} \tag{86}$$

$$\frac{\partial h_n}{\partial g} = -\frac{\partial v_n}{\partial g}(w_{n-1} - u_n - v_n)\frac{1}{2g_n} = -\frac{t_n^2}{2} \tag{87}$$

$$\frac{\partial h_1}{\partial w_0} = (w_0 - u_1 - v_1)\frac{1}{g_1} = -t_1 \tag{88}$$

$$A_{nn} = \frac{g_n^*}{w_n} \tag{89}$$

$$C_{nn} = \frac{\partial v_n}{\partial u_n}\frac{v_n(1 - g_n^*/g_n)}{w_n} - \frac{g_n^*}{g_n}\frac{v_n}{w_n} \tag{90}$$

$$\begin{aligned} D_{nn} &= \frac{\partial v_n}{\partial F_n}\frac{g_n^*/2g_n(w_{n-1} - u_n) + v_n(1 - g_n^*/2g_n)}{w_n} \\ &\qquad + \frac{g_n^*(s_n - h_n)}{w_n\, F_n} \end{aligned} \tag{91}$$

$$E_{nn} = \frac{\partial v_n}{\partial g_n}\frac{g_n^*/2g_n(w_{n-1} - u_n) + v_n(1 - g_n/2g_n)}{w_n} \tag{92}$$

$$B_{11} = \frac{\partial v_1}{\partial w_0}\frac{g_1^*/g_1(w_0 - u_1) + v_1(1 - g_1/g_1^*)}{w_1}. \tag{93}$$

Introducing $g_n = g_n^*$ as well as equations (81) throug (84) and using with (91) also equation (18) and th following expression:

$$\begin{aligned} 2g_n(s_n - h_n) &= \frac{2g_n\, M_0\, c}{F}\, u_n + v_n^2 - 2c \\ &\qquad \cdot [v_n - (w_{n-1} - u_n)] - (w_{n-1} - u_n \end{aligned}$$

equations (89) through (93) evolve into the fin expressions for the error coefficients of the very las period:

$$\boxed{\begin{aligned} &A_{nn} = \frac{g_n}{w_n} \qquad C_{nn} = -\frac{v_n}{w_n} && (94), (95 \\ &D_{nn} = \frac{1}{F_n}\frac{g_n\, M_{0n}\, c^2}{F_n\, w_n}\left[\frac{u_n}{c} - 1 + e^{-u_n/c}\right] && (96 \\ &E_{nn} = -\frac{1}{2g_n\, w_n}\left[(w_{n-1} - u_n)^2 - v_n^2\right] && (97 \\ &B_{11} = \frac{w_0 - u_1}{w_1} && (98 \end{aligned}}$$

Equations (94) through (97) also hold for A_{ii}, C_{ii}, D_{ii} and E_{ii} by proper replacement of indices.

Conclusions from (95) and (96):

$\equiv$ is symbolic for logic EQUIVALENCE

$\sim$ is symbolic for logic NEGATION

$\vee$ is symbolic for logic DISJUNCTION

$$(C_{nn} = 0) \equiv (v_n = 0) \vee (w_n = \infty) \tag{99}$$

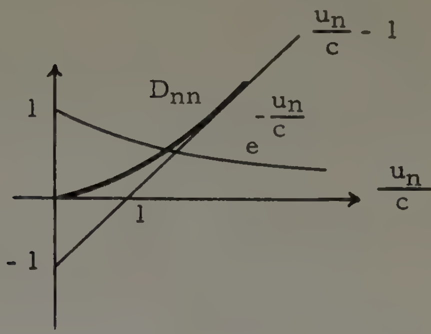
$w_n = \infty$ is impractical

$$(v_n = 0) \equiv \left[u_n = w_{n-1} + \frac{g_n M_{0_n} c}{F_n} (1 - e^{-u_n/c}) \right] \equiv [u_n = w_{n-1} + g_n t_n] \tag{100}$$

$$(C_{nn} = 0) \equiv \left[u_n = w_{n-1} + \frac{g_n M_{0_n} c}{F_n} (1 - e^{-u_n/c}) \right] \equiv [u_n = w_{n-1} + g_n t_n] \tag{101}$$

$$(D_{nn} = 0) \equiv \left[\frac{u_n}{c} - 1 + e^{-u_n/c} = 0 \right] \vee (w_n = \infty) \tag{102}$$

$$\left[\frac{u_n}{c} - 1 + e^{-u_n/c} = 0 \right] \equiv \left(\frac{u_n}{c} = 0 \right) \tag{103}$$



Thus for all practical reasons:

$$\sim (D_{nn} = 0). \tag{104}$$

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Technical Notes

Comments on "Anisotropy of Escape Velocity from the Moon"[1]

Howard S. London[2]

In a recent article appearing in this journal (1), Gold concludes that a difference of 30% exists between the escape velocities on the near and far sides of the Moon due to the gravitational influence of the Earth. Calculations based on Jacobi's integral of the restricted three-body problem do not support this conclusion; an anisotropy does exist but it is no more than one-sixth of one percent.

Mr. Gold compares the minimum velocity required at the near surface of the Moon to reach the libration point lying on the Earth-Moon axis on the near side of the Moon (a libration point being defined as a point where a body will remain in equilibrium if initially placed there at rest) with the velocity required on the far surface of the Moon to escape to zero velocity at infinity. Such a comparison is meaningless, since a libration point can be reached equally well from either side of the Moon. The proper comparison therefore is between the minimum velocities required to reach this point from the near and far surfaces, respectively. These velocities are easily calculated from Jacobi's well-known integral of the restricted three-body problem (2). Assuming that the mass of the Earth is 81.45 times the mass of the Moon, the diameter of the Moon is 3478 km, and that the Earth and Moon are separated by 384,400 km, it is found that the required velocities on the near and far surfaces of the Moon are identical to six significant figures (= 2.31946 km/sec).

Even if one chooses instead to compare the velocity required on the near surface of the Moon to reach the libration point which is on the near side, and the velocity required on the far surface to reach the libration point which is on the far side, the difference is quite small. These velocities are:

$$V_{\text{near surface}} = 2.31946 \text{ km/sec}$$

$$V_{\text{far surface}} = 2.32312 \text{ km/sec}$$

Thus, even on this basis of comparison the anisotropy is only about one-sixth of one percent rather than 30% as estimated by Gold.

Furthermore, the approximate formula Eq. (2b) of (1), derived by Gold in (3), gives $V_{\text{near surface}} = 1.889$ km/sec, which is 18.5% lower than the value of 2.319 km/sec as determined from Jacobi's integral. This discrepancy is due to the fact that the analysis in (3) is based upon a model of the Earth-Moon system in which the Earth and Moon are stationary; such a model is inadequate since it neglects the Coriolis and centrifugal forces due to the rotation of the Earth-Moon axis.

References

(1) Gold, L., Anisotropy of Escape Velocity from the Moon, the Lunar Atmosphere and the Origin of Craters. Journal of the Astronautical Sciences, **7** (1960), pp. 23–24.

(2) Moulton, F. R., Celestial Mechanics, the MacMill[an] Company, 1914, pp. 237–294.

(3) Gold, L., Earth-Moon Rocket Trajectories. Journal of t[he] Franklin Institute, **266**, 1 (1958).

[1] Received June 1960.

[2] United Aircraft Corporation Research Laboratories, East Hartford, Connecticut.

Geodetic Sub-Latitude and Altitude of a Space Vehicle[1]

Robert H. Gersten[2, 3]

This technical note outlines a procedure for determini[ng] the geodetic sub-latitude and altitude of a space vehicle fro[m] its geocentric position. The geocentric and geodetic su[b-]latitudes of the vehicle, denoted by ϕ' and ϕ respectivel[y] are related by the conventional equations (see Referenc[es] 1 and 2 at end of article):

$$(C + H) \cos \phi = r \cos \phi' \quad (1)$$

$$(S + H) \sin \phi = r \sin \phi', \quad (2)$$

where, utilizing a_e, e, and f to represent the Earth's equ[a]torial radius, eccentricity, and flattening and r to represe[nt] the separation between the vehicle and geocenter,

$$C = \frac{a_e}{(1 - e^2 \sin^2 \phi)^{1/2}}, \qquad S = C(1 - e^2) \qquad e^2 = 2f - f^2 \quad (3)$$

and where H is the height above the geoid expressed in th[e] same units as a_e (see Fig. 1). Multiplying Equation (1) an[d]

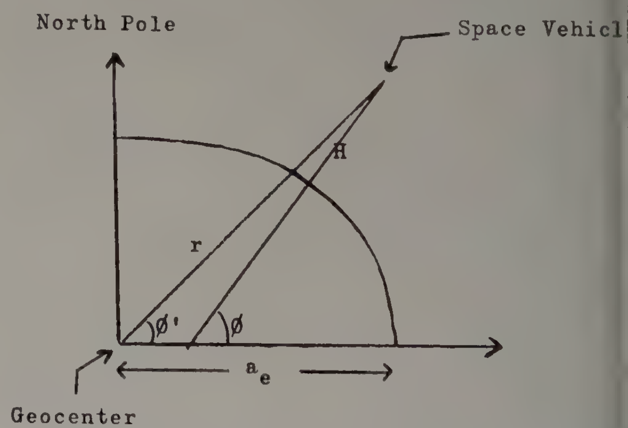

Fig. 1. Meridonal cross section of the Earth showing th[e] vehicle's geocentric distance r and geocentric and geodet[ic] sub-latitudes ϕ' and ϕ.

Equation (2) by $\sin \phi$ and $\cos \phi$ respectively, and subtracti[ng] yields:

$$(C - S) \sin \phi \cos \phi = r \sin (\phi - \phi'). \quad (4)$$

[1] Received October 1960.

[2] Senior Scientist, Astrodynamics Department.

[3] Norair Division, Northrop Corporation, Hawthorne, Cali[fornia].

m Equation (3)

$$C - S = e^2 C = \frac{a_e e^2}{(1 - e^2 \sin^2 \phi)^{1/2}}.$$

forming a binomial expansion on the above

$$C - S = a_e e^2 (1 + \tfrac{1}{2} e^2 \sin^2 \phi)$$

ere powers of e greater than the fourth are truncated. ostitution of the above into Equation (4) yields:

$$\sin(\phi - \phi') = \epsilon \sin\phi \cos\phi (1 + \tfrac{1}{2} e^2 \sin^2 \phi) \qquad (5)$$

ere

$$\epsilon = \frac{a_e e^2}{r}.$$

order to maintain all terms through the e^4 terms in Equa- (5), it is only necessary to find expressions for $\sin\phi$ and ϕ as a function of ϕ' and e, which are expanded through e^2 term (since the right side of the Equation is multiplied $\left(\epsilon = \frac{a_e e^2}{r}\right)$. This is accomplished in the manner indicated ow.

Substituting Equation (5) into

$$\cos(\phi - \phi') = [1 - \sin^2(\phi - \phi')]^{1/2}$$

l expanding, yields

$$\cos(\phi - \phi') = 1 - \tfrac{1}{2}\epsilon^2 \sin^2\phi + \tfrac{1}{2}\epsilon^2 \sin^4\phi. \qquad (6)$$

ostitution of Equations (5 and 6) into the trigonometric ntities:

$$\phi = \sin[\phi' + (\phi - \phi')]$$

$$= \sin\phi' \cos(\phi - \phi') + \cos\phi' \sin(\phi - \phi')$$

$$\phi = \cos[\phi' + (\phi - \phi')]$$

$$= \cos\phi' \cos(\phi - \phi') - \sin\phi' \sin(\phi - \phi')$$

es

$$\sin\phi = \sin\phi' + \cos\phi'(\epsilon \sin\phi \cos\phi)$$

$$\cos\phi = \cos\phi' - \sin\phi'(\epsilon \sin\phi \cos\phi).$$

previously explained, it is unnecessary to keep terms taining e raised to higher than the second power (or ϵ sed to higher than the first power) in the above in order to maintain accuracy through the e^4 term. Substitution of the expressions for $\sin\phi$ and $\cos\phi$ above into the $\sin\phi$ and $\cos\phi$ terms on the right side of the above yields:

$$\sin\phi = \sin\phi'(1 + \epsilon\cos^2\phi') \qquad (7a)$$

$$\cos\phi = \cos\phi'(1 - \epsilon\sin^2\phi'). \qquad (7b)$$

Now substitution of Equations (7) into Equation (5) yields:

$$\sin(\phi - \phi') = \epsilon \sin\phi' \cos\phi'[1 + \epsilon - (2\epsilon - \tfrac{1}{2}e^2)\sin^2\phi']$$

where

$$\epsilon = \frac{a_e e^2}{r}.$$

The above may be expressed in terms of flattening by:

$$\sin(\phi - \phi') = \frac{a_e}{r}\left[f \sin 2\phi' + f^2 \sin 4\phi' \left(\frac{a_e}{r} - \frac{1}{4}\right)\right].$$

Similarly, eliminating C and S from Equations (1, 2, and 3), the geodetic altitude, H, may be obtained as

$$H = r - a_e[1 - \tfrac{1}{2}e^2(1 + \epsilon)\sin^2\phi' + \tfrac{1}{2}e^2(\epsilon - \tfrac{1}{4}e^2)\sin^4\phi']$$

where again

$$\epsilon = \frac{a_e e^2}{r}$$

or, in terms of the Earth's flattening,

$$H = r - a_e\left[1 - f\sin^2\phi' - \frac{f^2}{2}\sin^2 2\phi'\left(\frac{a_e}{r} - \frac{1}{4}\right)\right].$$

See pages 166 and 167 of (2).

Acknowledgment

This work was stimulated by a problem proposed in an astrodynamics course at the University of California, Los Angeles in March, 1960. A number of exceedingly important suggestions and improvements made by Dr. Samuel Herrick are greatfully acknowledged.

Reference

(1) HERRICK, S., *Astrodynamics* Van Nostrand, 1961.
(2) BAKER, R. M. L., JR. AND MAKEMSON, M. W., *An Introduction to Astrodynamics*, Academic Press, 1960, Chapter 4.

Letters to the Editor

Comments on "The Influence of Heart Action and the Circulation of Blood on Manned Satellite Attitude Control,"

Gentlemen: October 28, 1960

In the Fall 1960 (Vol. VII, No. 3) issue of the *Journal of the Astronautical Sciences* the paper by Lawrence R. Zeitlin entitled "The Influence of Heart Action and the Circulation of the Blood on Manned Satellite Attitude Control" contains a fundamental error in mechanics that leads to a false conclusion about the requirements of the satellite attitude control system.

Specifically, Mr. Zeitlin first calculates the effective angular momentum of the blood circuit of a man. He then calculates the angular momentum of the satellite for any angular velocity. He then, erroneously, equates these two angular momenta and solves for the angular rate of the satellite. Furthermore, he then states, erroneously, "Over a long period of time, a significant amount of energy would be required from the attitude control system to counteract this torque", produced by the "reaction to the angular momentum of the astronaut's circulatory system".

Assuming that the satellite and the enclosed astronaut have been initially stabilized, the only time the satellite will change its angular momentum (start rotating) in an inertial reference frame is when the angular momentum of the astronaut's blood *changes* in the inertial reference frame, and then the change in the satellite momentum will be such as to keep the angular momentum of the satellite plus the angular momentum of the astronaut's blood constant in the inertial reference frame. If the astronaut is kept relatively motionless in the satellite, as postulated by Mr. Zeitlin, the only justification for equating the two angular momenta is to assume that the astronaut's blood has stopped circulating. Under this condition the satellite will indeed begin to rotate (with angular momentum equal to the angular momentum of the astronaut's blood before it stopped circulating), however, an angular momentum *impulse* from the attitude control system can stop this rotation. No further energy expenditure from the control system will be required to keep the satellite from starting to rotate again.

Considering a more general situation (also more palatable to the astronaut), assume that the astronaut is free to move about inside the satellite. He is thus able to change the direction (in the inertial reference frame) of his blood's angular momentum. The satellite will rotate in the inertial reference frame so as to maintain constant total angular momentum of the system (satellite plus astronaut). Note that angular momentum is a vector quantity and that a constant vector has constant magnitude *and* direction. If the astronaut moves from one orientation to another the satellite may start to rotate but an angular momentum *impulse* from the attitude control system can stop this rotation, and it will *not start again* until the astronaut moves again. If the astronaut moves around considerably the total energy involved in all the momenta impulses may become significant if a non-conservative attitude control mechanism is used, such as reaction gas jets. A highly suitable conservative attitude control mechanism is available however, in the form of reaction flywheels. These reaction wheels provide a third component of angular momentum in the overall system that can balance the astronaut's blood momentum changes and thus prevent the satellite from rotating when the astronaut change[s] orientation in the satellite.

HENRY M. MUSAL, JR.
Senior Engineer, Research Laborator[y],
Bendix Systems Division,
493 East 31st. Street,
Chicago 16, Illinois

November 30, 196[0]

Gentlemen:

Mr. Musal is perfectly correct in stating that an error wa[s] made in the original paper. This was an error in computatio[n] however, rather than a "fundamental error in mechanics." We simply did not use the same assumptions regarding initia[l] stabilization of the satellite. The error was corrected in [a] revised copy of the paper, which unfortunately, did no[t] arrive until after the publication deadline. The correcte[d] material should read:

"The heart of a normal man pumps between 4 and 6 Kg[.] of blood per minute; or about 7200 Kg. in a 24 hour da[y] (1). At any time, he has the equivalent of a fluid flywhee[l] within his body with a mass of approximately 5 Kg., [a] radius of 5 cm., and rotating once a minute. This flywhee[l] has a small, but finite amount of angular momentum[,] which, in accordance with the principle of conservation o[f] momentum, would exert a rotating torque on his bod[y] in the opposite direction. Since the blood flows clockwise[,] looking from the front, the torque tending to rotate th[e] body would be counterclockwise.

What would the effect of this rotating torque be on [a] small manned satellite?

The angular momentum of a body is defined as th[e] product of its rotary inertia and the angular velocit[y] around a given axis. For a flywheel with its weight concentrated in the rim, the moment of inertia $(I) = MR^2$ where M is the mass and R is the radius. Hence the angula[r] momentum of the equivalent blood circuit is:

$$= MR^2 W = \frac{(5 \times 10^3)(25)(6.28)}{60}$$

$$= 1.31 \times 10^4 \text{ gm cm}^2 \text{ radians/second.}$$

For a 1000 Kg. satellite, of approximately the shap[e] of the Mercury vehicle, the moment of inertial along it[s] axis is that of a right cone of altitude H radius of base R, and mass M. That is:

$$I = M \frac{3}{10} R^2$$

For the Mercury vehicle,

$$M = 1000 \text{ Kg}$$

$$R = 100 \text{ cm}$$

$$H = 300 \text{ cm}$$

Hence:

$$I = 10^6 \times 10^4(.3) = 3 \times 10^9 \text{ gm cm}^2.$$

According to the law of conservation of momentum, the angular momentums of the two rotating masses mus[t]

be equal. Hence:

$$\underset{\text{Ang. momentum of blood}}{1.31 \times 10^4 \text{ gm cm}^2 \text{ rad./sec.}} \qquad \underset{\text{Ang. momentum of vehicle}}{3 \times 10^9 \text{ gm cm}^2 W}$$

$$W = 4.36 \times 10^{-6} \text{ rad./sec.}$$
or
$$2.5 \times 10^4 \text{ deg./sec.}$$

In a 24 hour day, a 1000 Kg. satellite would rotate 21° 40′ about its axis simply in reaction to the angular momentum of the astronaut's circulatory system.

I fully agree that, given a state of initial stabilization, the nly time the satellite will change its angular momentum in n inertial reference frame is when the angular momentum f the astronaut's blood also changes in the inertial reference ame. Mr. Musal gives a backhanded justification for equatg momentii of the satellite and astronaut by stating that the astronaut's blood stopped circulating the satellite if abilized would begin to rotate with an angular momentum qual to that of the angular momentum of the astronaut's lood. Being of a more sanguine nature, I preferred to start ith a satellite-astronaut system of zero initial angular omentum and determine the maximum rotation of the system attributable to the angular momentum of the astronaut's blood. No assumption of initial stabilization was made. If uncorrected, the rotation proposed by Mr. Musal as a function of the death of the astronaut would be equal to the rotation computed above, but in the opposite direction. It's just another view of the elephant.

As for the more general case, the angular momentum of the astronaut's blood changes from moment to moment. States of sleep, excitement, digestion, deep thinking, and activity all produce changes in blood pressure and volume of circulation which can be reflected in changes of angular momentum. Try taking your pulse after resting, eating, or chasing your secretary around the desk. Pulse (and angular momentum) may vary as much as 3:1 under these conditions. The attitude control system, whether conservative or non-conservative, would have to correct continually for these angular momentum changes if precise attitude control is required since these changes are not of a discrete nature.

By the way, if the astronaut can orient his seat in any direction, he can serve as his own reaction wheel.

Lawrence R. Zeitlin
Dunlap and Associates, Inc.
355 W. Olive Ave.
Sunnyvale, California

Book Review

Review of "Interplanetary Communications", Chapter 2 f *Advances in Space Science*, Vol. 1, Academic Press, New ork and London, 1959.

This is an excellent survey of the subject by Dr. J. R. ierce and Mr. C. C. Cutler of Bell Telephone Laboratories. is natural that the Bell Telephone System would explore e technical and economic feasibility of using space to lve its long distance communication problems such as the ansatlantic telephone problem. The highlights of a high-ade study of low and high altitude communication satellites gether with pertinent charts and formulas are most welme.

Besides the fine treatment of the use of space vehicles to crease the number of channels and reliability while greatly ducing the cost of Earth communications, this Chapter so considers the problems of interplanetary and interstellar communication. The treatment is from the modern sophisticated information theoretic and practical point of view. Modulation methods are compared with the ideal, but as yet unrealized, technical optimum. Technical-economic optima for choice of transmission frequency are given. It is interesting to see that the granularity difficulties which would arise for communication at X-ray frequencies are not serious for a wavelength of 1 cm. Here the authors estimate 80 quanta per cycle bandwidth.

This summary of ideas on space communication, with its sprinkling of profound remarks on what is important, is highly recommended to both the expert and the layman.

John L. Barnes
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1007 Broxton, Los Angeles 24, California

Format of Technical Papers for AAS Journal

The Editors will appreciate the cooperation of ıthors in using the following directions for the prepara-on of manuscripts. These directions have been com-led with a view toward eliminating unnecessary cor-espondence, avoiding the return of papers for changes, ıd reducing the charges made for "author's correc-ons."

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The level of subscripts, exponents, subscripts to sub-ripts, and exponents in exponents should be clearly dicated.

Greek Alphabet

Α	α	alpha	(a)	Ν	ν	nu	(n)
Β	β	beta	(b)	Ξ	ξ	xi	(x)
Γ	γ	gamma	(g)	Ο	ο	omicron	(ŏ)
Δ	δ	delta	(d)	Π	π	pi	(p)
Ε	ϵ	epsilon	(ĕ)	Ρ	ρ	rho	(r)
Ζ	ζ	zeta	(z)	Σ	σς	sigma	(s)
Η	η	eta	(ē)	Τ	τ	tau	(t)
Θ	θ	theta	(th)	Υ	υ	upsilon	(u)
Ι	ι	iota	(i)	Φ	$\phi\varphi$	phi	(ph)
Κ	κ	kappa	(k)	Χ	χ	chi	(ch)
Λ	λ	lambda	(l)	Ψ	ψ	psi	(ps)
Μ	μ	mu	(m)	Ω	ω	omega	(ō)

ϵ (epsilon) = strain

σ, ς (sigma) = stress

τ (tau) = shear stress

μ (mu) = micro

μ (mu) = Poisson's Ratio

Complicated exponents and subscripts should be avoided when possible to represent by a special symbol.

Fractions in the body of the text and fractions occurring in the numerators or denominators of fractions should be written with the solidus. Thus:

$$\frac{\cos(\pi x/2b)}{\cos(\pi\alpha/2b)}$$

is the preferred usage.

The intended grouping of handwritten formulas can be made clear by slight variations in spacing, but this procedure is not acceptable in printed formulas. To avoid misunderstanding, the order of symbols should therefore be carefully considered. Thus:

$(a + bx)\cos t$ is preferable to $\cos t\,(a + bx)$

In handwritten formulas the size of parentheses, brackets and braces can vary more widely than in print. Particular attention should therefore be paid to the proper use of braces, brackets, and parentheses (which should be used in this order). Thus:

$$\{[a + (b + cx)^n] \cos ky\}^2$$

is required rather than $((a + (b + cx)^n) \cos ky)^2$.

Equations are numbered and referred to in text as (15).

Bibliography

References should be grouped together in a bibliography at the end of the manuscript. References to the bibliography should be made by numerals between square brackets [4].

The following examples show the approved arrangements:

for books—[1] HUNSAKER, J. C. and RIGHTMIRE, B. S., *Engineering Applications of Fluid Mechanics*, McGraw-Hill Book Co., New York, 1st ed., 1947, p. 397.

for periodicals—[2] Singer, S. F., "Artificial Modification of the Earth's Radiation Belt," *J. Astronaut. Sci.*, *6* (1959), 1–10.